Supplementary Materials for the paper "Magnetic helices as metastable states of finite XY ferromagnetic chains: an analytical study" by A. P. Popov and M. G. Pini

The ground state of model (1) corresponds to ferromagnetic ordering of spins because the state with zero magnitude of all orientation angles corresponds to global minimum of the function $E(\theta_1, \theta_2, ..., \theta_N)$. Here we search for additional minima of the function (1) that correspond to metastable states of a system. Minimization of energy (1) gives rise to a system of $N$ equations for the orientation angles $\theta_n$ ($n=1,2,3,....,N$).

$$\frac{\partial E}{\partial \theta_1} = 0 \quad \Rightarrow \quad \sin(\theta_1 - \theta_2) + \sin(\theta_1 - \theta_0) = 0 \quad (n=1) \quad (2.1)$$

$$\frac{\partial E}{\partial \theta_n} = 0 \quad \Rightarrow \quad \sin(\theta_n - \theta_{n+1}) + \sin(\theta_n - \theta_{n-1}) = 0 \quad (2 \leq n \leq N-1) \quad (2.2)$$

$$\frac{\partial E}{\partial \theta_N} = 0 \quad \Rightarrow \quad \sin(\theta_N - \theta_{N+1}) + \sin(\theta_N - \theta_{N-1}) = 0 \quad (n=N) \quad (2.3)$$

$$\theta_0 = 0, \quad \theta_{N+1} = 0$$

Left part of equation (2.2) can be rewritten in a form of a product of two functions

$$\sin(\theta_n - \theta_{n+1}) + \sin(\theta_n - \theta_{n-1}) = 2 \sin\left(2\theta_n - \frac{\theta_{n+1} - \theta_{n-1}}{2}\right) \cos\left(\frac{\theta_{n+1} - \theta_{n-1}}{2}\right) = 0. \quad (3)$$

It follows from equation (3) that it can be satisfied if any of two following equations is satisfied.

$$\sin\left(2\theta_n - \frac{\theta_{n+1} - \theta_{n-1}}{2}\right) = 0 \quad \Rightarrow \quad \theta_{n+1} + \theta_{n-1} - 2\theta_n = 2\pi k \quad k \in Z \quad (3.1)$$

$$\cos\left(\frac{\theta_{n+1} - \theta_{n-1}}{2}\right) = 0 \quad \Rightarrow \quad \theta_{n+1} - \theta_{n-1} = \pi + 2\pi k \quad k \in Z \quad (3.2)$$

**Equation (3.1).**

We believe that angle between spins of neighbor atoms does not exceed $\pi$. If it is equal to zero or $\pi$ then the collinear ferromagnetic (FM) or antiferromagnetic (AF) ordering takes place. These are particular cases of more general spiral state. FM ordering corresponds to stable ground state and its energy is minimal. AF ordering corresponds to unstable state because the integral of exchange interaction in this system is positive that is it corresponds to FM ordering. The energy of AF state is maximal. Here we consider only non collinear magnetic structures. The restriction on the left side of (3.1) follows from the following obvious inequalities (the angle between vectors is sharp):
We add these double inequalities to each other and get the following restriction
\[-2\pi < \theta_{n+1} + \theta_{n-1} - 2\theta_n < 2\pi.\]
It follows from this formula that one should put $k = 0$ in the right side of equation (3.1). Then one arrives at the following recurrence equation.
\[\theta_{n+1} - 2\theta_n + \theta_{n-1} = 0.\]  
(4)

The succession
\[\theta_n = \theta_i + \alpha(n-1)\]  
(5)
satisfies this recurrence equation for any values of parameters $\theta_i$ and $\alpha$. Eq. (5) describes simple spiral structure in which the spin is rotated by the same angle $\alpha$ when moving along the chain from one atom to the neighboring atom. Magnitudes of parameters $\theta_i$ and $\alpha$ can be determined from the equations (2.1) and (2.3), which play the role of the boundary conditions. Model (1) has a symmetry between the clockwise and counterclockwise senses of rotation of spiral structure. So without the loss of generality we can restrict the search of the spiral solution of the equation (2.2) corresponding to a right-handed helix, that is, we assume that $0 < \alpha < \pi$.

With account of equalities $\theta_0 = 0, \quad \theta_{N+1} = 0$ and formula (5) equations (2.1) and (2.3) can be written in a more simple form
\[
\begin{cases}
\sin \theta_i = \sin \alpha \\
\sin (\theta_i + \alpha(N-1)) = -\sin \alpha
\end{cases}
\]  
(6.1)
(6.2)

The equation (6.1) is equivalent to the following set of equations
\[
\begin{cases}
\theta_i = \alpha + 2\pi k, \quad k \in Z \\
\theta_i = \pi - \alpha + 2\pi l, \quad l \in Z
\end{cases}
\]
Parameters $\theta_i$ and $\alpha$ belong to the interval $(0, \pi)$ because the difference between them cannot exceed $\pi$. Because of that here we put $k = 0, \quad l = 0$ and write
\[
\begin{cases}
\theta_i = \alpha \\
\theta_i = \pi - \alpha
\end{cases}
\]  
(7.1)
(7.2)
The equation (6.2) is equivalent to the following set of equations
Each straight lines (7.1) and (7.2) cross each straight line (8.1) and (8.2) inside the square $0 < \alpha < \pi$ only one time. Hence in this area system of equations (6.1) and (6.2) has four solutions which we need to get. Below we consider that equations (7.1) and (7.2) determine first and second branches of solution whereas equations (8.1) and (8.2) determine first and second kind of solution. Below the first upper index in a round brackets numbers the branch of solution, second index in round brackets numbers the kind of solution.

### 1.1 First branch of solution, first kind of solution

**System of equations (7.1)+(8.1)**

\[
\begin{align*}
\theta_i &= \alpha \\
\theta_i &= \pi - \alpha (N-2) + 2\pi n
\end{align*}
\]

\[
\alpha^{(1)} = \frac{2\pi m}{N+1}, \quad 0 \leq m \leq \frac{N+1}{2}.
\]

Energy levels of these spiral magnetic structures are determined by formula

\[
\varepsilon^{(1)}(m) = 2(N+1)\sin^2\left(\frac{\pi m}{N+1}\right).
\]

### 1.2 First branch of solution, second kind of solution

**System of equations (7.1)+(8.2)**

\[
\begin{align*}
\theta_i &= \alpha \\
\theta_i &= \pi - \alpha (N-2) + 2\pi n
\end{align*}
\]

\[
\alpha^{(1)} = \frac{\pi (2n-1)}{N-1}, \quad 1 \leq n \leq \frac{N}{2}.
\]

Energy levels of these spiral magnetic structures are determined by formula

\[
\varepsilon^{(1)}(n) = 2 + 2(N-1)\sin^2\left(\frac{\pi (2n-1)}{2(N-1)}\right).
\]

### 2.1 Second branch of solution, first kind of solution

**System of equations (7.2)+(8.1)**

\[
\begin{align*}
\theta_i &= \pi - \alpha \\
\theta_i &= -\alpha N + 2\pi m
\end{align*}
\]

\[
\alpha^{(2)} = \frac{\pi (2m-1)}{N-1}, \quad 1 \leq m \leq \frac{N}{2}.
\]

Energy levels of these spiral magnetic structures are determined by formula

\[
\varepsilon^{(2)}(m) = 2 + 2(N-1)\sin^2\left(\frac{\pi (2m-1)}{2(N-1)}\right).
\]
2.2 Second branch of solution, second kind of solution

System of equations (7.2)+(8.2)

\[
\begin{align*}
\theta_1 &= \pi - \alpha \\
\theta_1 &= \pi - \alpha (N-2) + 2\pi n,
\end{align*}
\]

\[\Rightarrow \quad \alpha^{(22)} = \frac{2\pi n}{N-3}, \quad 0 \leq n \leq \frac{N-3}{2}.
\]

(12.1)

Energy levels of these spiral magnetic structures are determined by formula

\[
\varepsilon^{(22)}(m) = 4 + 2(N-3)\sin^2\left(\frac{\pi n}{N-3}\right).
\]

(12.2)

One can get convinced that the following relations between the angles \(\theta_1\) and \(\theta_N\) are valid for various branches and kinds

\[
\begin{align*}
\theta_1^{(11)} &= \alpha^{(11)}, & \cos \theta_1^{(11)} &= \cos \alpha^{(11)}, & \cos \theta_N^{(11)} &= \cos \alpha^{(11)}, & \cos \theta_1^{(11)} &= + \cos \theta_1^{(11)}, \\
\theta_1^{(12)} &= \alpha^{(12)}, & \cos \theta_1^{(12)} &= \cos \alpha^{(12)}, & \cos \theta_N^{(12)} &= - \cos \alpha^{(12)}, & \cos \theta_1^{(12)} &= - \cos \theta_1^{(12)}, \\
\theta_1^{(21)} &= \pi - \alpha^{(21)}, & \cos \theta_1^{(21)} &= - \cos \alpha^{(21)}, & \cos \theta_N^{(21)} &= + \cos \alpha^{(21)}, & \cos \theta_1^{(21)} &= - \cos \theta_1^{(21)}, \\
\theta_1^{(22)} &= \pi - \alpha^{(22)}, & \cos \theta_1^{(22)} &= - \cos \alpha^{(22)}, & \cos \theta_N^{(22)} &= - \cos \theta_1^{(22)}.
\end{align*}
\]

(13)

Stability of spiral states

In this section we determine which solutions found in the previous section correspond to local minima of energy (1), and thus correspond to metastable spiral structures. To do this we assume the chain to be in one of the spiral states determined by Eqs. (9-12). This state is determined by the index of branch \(i\), the kind of solution \(p\) and mode number \(m\). Then one should deviate each orientation angle \(\theta_n\) from its equilibrium value \(\theta_n^{(0)}(i, p, m) + \nu_n\), where \(\nu_n\) is deviation and analyze the sign of increment in energy. The increment in the energy (1) in quadratic approximation can be written as:

\[
\Delta \varepsilon = \frac{1}{2} A_{mm} \nu_n \nu_n'.
\]

(14)

Here \(A\) – square matrix of second derivatives of energy (1) \(A_{mm} = \partial^2 \varepsilon / \partial \theta_n \partial \theta_m\) named Hessian, \(A\) is symmetrical three-diagonal \((N \times N)\) matrix with real matrix elements

\[
A_{mm} = \cos (\theta_n - \theta_{n+1}) + \cos (\theta_n - \theta_{n-1}), \quad n = 1, 2, \ldots, N, \quad \theta_0 = 0, \quad \theta_{N+1} = 0.
\]

(15)

In the particular case of simple spiral structure when orientation angles are determined by formula \(\theta_n = \theta_1 + \alpha(n-1)\) matrix \(A\) can be written as
Energy is minimal at the point \( (\theta_1^{(0)}, \theta_2^{(0)}, \ldots, \theta_N^{(0)}) \) if the matrix that determines this quadratic form is positive definite. There are several criteria for positive definiteness of the matrix. One of them consists in the fact that all the eigenvalues of the matrix must be positive. It is convenient to use when it is easy to obtain analytical expressions for the eigenvalues. There is also the criterion of Sylvester, according to which the matrix is positive definite, if all angular minors of this matrix, including its determinant are positive. We here use both of these criteria.

1.1 First branch of solution, first kind of solution

\[
\alpha^{(1)} = \frac{2\pi m}{N+1}, \quad 0 \leq m \leq \frac{N+1}{2}.
\]

\[
\theta_1^{(1)} = \alpha^{(1)}, \quad \cos \theta_1^{(1)} = \cos \alpha^{(1)}, \quad \cos \theta_N^{(1)} = \cos \alpha^{(1)}, \quad \cos \theta_N^{(1)} = + \cos \theta_1^{(1)}
\]

In this case \( A \) that determines quadratic form is as follows

\[
A = \begin{pmatrix}
\cos(\alpha) + \cos(\theta_i) & -\cos(\alpha) & 0 & \ldots & 0 & 0 & 0 \\
-\cos(\alpha) & 2\cos(\alpha) & -\cos(\alpha) & \ldots & 0 & 0 & 0 \\
0 & -\cos(\alpha) & 2\cos(\alpha) & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 2\cos(\alpha) & -\cos(\alpha) & 0 \\
0 & 0 & 0 & \ldots & -\cos(\alpha) & 2\cos(\alpha) & -\cos(\alpha) \\
0 & 0 & 0 & \ldots & 0 & -\cos(\alpha) & \cos(\alpha) + \cos(\theta_N^{(0)})
\end{pmatrix}
\]  

(16)

Matrix \( A_{11} \) in Eq. (17) can be written as \( A_{11} = (2\cos \alpha) M_{11} \), where \( M_{11} \) is the following matrix

\[
M_{11} = \begin{pmatrix}
1 & -1/2 & 0 & \ldots & 0 & 0 & 0 \\
-1/2 & 1 & -1/2 & \ldots & 0 & 0 & 0 \\
0 & -1/2 & 1 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 1 & -1/2 & 0 \\
0 & 0 & 0 & \ldots & -1/2 & 1 & -1/2 \\
0 & 0 & 0 & \ldots & 0 & -1/2 & 1
\end{pmatrix}
\]  

(18)
Eigenvalues $\lambda_M$ of matrix $M_{11}$ and eigenvalues $\lambda_A$ of matrix $A_{11}$ are related by $\lambda_A = (2 \cos \alpha) \lambda_M$. One can find eigenvalues $\lambda_M$ of matrix $M_{11}$ if one solves equation

$$\det(M_{11} - \lambda_M I) = \det \begin{pmatrix} 1 - \lambda_M & -1/2 & 0 & \ldots & 0 & 0 & 0 \\ -1/2 & 1 - \lambda_M & -1/2 & \ldots & 0 & 0 & 0 \\ 0 & -1/2 & 1 - \lambda_M & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 - \lambda_M & -1/2 & 0 \\ 0 & 0 & 0 & \ldots & -1/2 & 1 - \lambda_M & -1/2 \\ 0 & 0 & 0 & \ldots & 0 & -1/2 & 1 - \lambda_M \end{pmatrix} = 0.$$

(19)

Matrix $M_{11}$ in (18) is a so-called oscillatory three-diagonal Jacobian matrix with real elements. It is known that all its eigenvectors form a basis in the $N$-dimensional space, and all eigenvalues are different and real. To begin with we search for the eigenvalues in the interval $0 \leq \lambda_M \leq 2$, i.e. $-1 \leq 1 - \lambda_M \leq 1$. Latter inequality means that one can use the following parametrization: $1 - \lambda_M = \cos \varphi$, $\varphi > 0$. It means that we shall search for the magnitude of parameter $\varphi$ instead of searching for parameter $\lambda_M$. Then Eq. (19) can be written as

$$\frac{\sin \varphi (N+1)}{2^N \sin \varphi} = 0 \rightarrow \varphi = \frac{\pi k}{N+1} \rightarrow 1 - \lambda_M \equiv \cos \varphi = \cos \frac{\pi k}{N+1}.$$

Consequently, eigenvalues $\lambda_M$ of matrix $M_{11}$ are equal to

$$\lambda_M = 1 - \cos \frac{\pi k}{N+1}, \quad k = 1, 2, \ldots, N$$

(20)

One can see that in the interval $0 \leq \lambda_M \leq 2$ we have found all $N$ eigenvalues. Because of that there is no need to search for $\lambda_M$ in any other intervals – it must be clear that they are absent in the other intervals. Therefore, based on this fact one has to conclude that all eigenvalues $\lambda_M$ of matrix $M_{11}$ are positive. Due to the relation $\lambda_A = (2 \cos \alpha) \lambda_M$ one has to conclude that eigenvalues $\lambda_A$ of matrix $A_{11}$ are equal to

$$\lambda_A = 2 \cos \alpha \left(1 - \cos \frac{\pi k}{N+1}\right), \quad k = 1, 2, \ldots, N.$$

(21)

It follows from this equation that eigenvalues $\lambda_A$ of matrix $A_{11}$ are positive if and only if $\cos \alpha$ is positive, i.e. $0 < \alpha < \pi/2$.

One can get this result using another method. Let us calculate corner minors of matrix $M_{11}$, determined by formula (17). In order to do this let us expand the determinant $d_n$ of matrix
We shall search for the solution of this recurrence equation in a form \( d_n \sim u^n \). Substituting \( d_n \) in this form in recurrence equation gives rise to quadratic equation for parameter \( u \)

\[
u^2 - u + \frac{1}{4} = 0 \quad \Rightarrow \quad u_{1,2} = \frac{1}{2}.
\]

By analogy with the solution of the differential equation in the case of multiple roots of the characteristic polynomial we write the general solution of the equation of recurrence equation in the following way

\[
d_n = A \left( \frac{1}{2} \right)^n + Bn \left( \frac{1}{2} \right)^n = \frac{A + Bn}{2^n}.
\]

Constants \( A \) and \( B \) are to be determined from the “initial conditions”

\[
\begin{align*}
    d_1 &= \frac{A + B}{2} = 1 \\
    d_2 &= \frac{A + 2B}{4} = \frac{3}{4} \\
\end{align*}
\]

\[
\Rightarrow \quad \begin{align*}
    A &= 1 \\
    B &= 1 \\
\end{align*}
\]

Finally, the expression for the determinant of the matrix \( M_{11} \) of dimension \( (n \times n) \) or that one and the same, for corner minor of the dimension \( (n \times n) \) of the matrix \( M_{11} \) can be written as

\[
d_n = \frac{1+n}{2^n}.
\]

One can see that all corner minors of matrix \( M_{11} \), including the determinant of matrix \( M_{11} \) are positive, that is matrix \( M_{11} \) is positive definite. Corner minors of matrix \( A_{11} \) can be easily obtained from corner minors of matrix \( M_{11} \)

\[
\text{Minor}(A_{11}, n) = (2 \cos \alpha)^n \frac{1+n}{2^n}.
\]

It follows from this expression that all minors of matrix \( A_{11} \) can be positive if and only if \( \cos \alpha > 0 \), i.e. when \( 0 < \alpha < \pi/2 \). Therefore, we came to the same conclusion in two different ways.

The obtained restriction on the angle \( \alpha \) from above, \( \alpha < \pi/2 \), is easy to understand referring to mean field theory. If \( \alpha < \pi/2 \) then the angle between spins \( S_{n-1} \) and \( S_{n+1} \), equal to
2\alpha$, is less than $\pi$. In this case spin $S_n$ is oriented along the sum of vectors $S_{n-1}$ and $S_{n+1}$. Obviously this state of spin $S_n$ is stable. It corresponds to minimum of the energy of exchange interaction of $S_n$ with neighbor spins $S_{n-1}$ and $S_{n+1}$. If however $\alpha > \pi/2$, then the angle between spins $S_{n-1}$ and $S_{n+1}$, equal to $2\alpha$ exceeds $\pi$. It means that mean field, which is parallel to sum of vectors $S_{n-1}$ and $S_{n+1}$ and vector $S_n$ are oriented in the opposite directions. Obviously this state of spin $S_n$ is unstable. It corresponds to maximum of energy of exchange interaction of spin $S_n$ with nearest neighbor spins $S_{n-1}$ and $S_{n+1}$.

1.2 First branch of solution, second kind of solution

$$\alpha^{(12)} = \frac{\pi(2n-1)}{N-1}, \quad 1 \leq n \leq \frac{N}{2}$$

$$\theta^{(12)}_i = \alpha^{(12)}, \quad \cos \theta^{(12)}_i = \cos \alpha^{(12)}, \quad \cos \theta^{(12)}_N = -\cos \alpha^{(12)}, \quad \cos \theta^{(12)}_N = -\cos \theta^{(12)}_i$$

Using these relations one can rewrite matrix $A$, Eq. (16), in the following way

$$A^{(12)} = \begin{pmatrix}
2\cos(\alpha) & -\cos(\alpha) & 0 & \ldots & 0 & 0 & 0 \\
-\cos(\alpha) & 2\cos(\alpha) & -\cos(\alpha) & \ldots & 0 & 0 & 0 \\
0 & -\cos(\alpha) & 2\cos(\alpha) & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 2\cos(\alpha) & -\cos(\alpha) & 0 \\
0 & 0 & 0 & \ldots & -\cos(\alpha) & 2\cos(\alpha) & -\cos(\alpha) \\
0 & 0 & 0 & \ldots & 0 & -\cos(\alpha) & 0
\end{pmatrix}. \quad (24)$$

Corner minor of $(2 \times 2)$ matrix in the right bottom corner of this matrix is equal to

$$d_2 = \begin{vmatrix}
2\cos(\alpha) & -\cos(\alpha) \\
-\cos(\alpha) & 0
\end{vmatrix} = -\cos^2(\alpha)$$

Obviously, it cannot be positive. It follows from this fact that spiral magnetic structure that corresponds to the second kind solution of the first branch is always unstable.

This result can be obtained with the help of analyses of the sign of minimal eigenvalue of matrix (24). One can demonstrate that matrix $A^{(12)}$ always has negative eigenvalue. However in the case considered the application of this method appears to be cumbersome and long. Because of that we put it in Appendix A.

2.1 Second branch of solution, first kind of solution

$$\alpha^{(21)} = \frac{\pi(2m-1)}{N-1}, \quad 1 \leq m \leq \frac{N}{2}.$$
$$\theta_i^{(21)} = \pi - \alpha^{(21)}, \quad \cos \theta_i^{(21)} = - \cos \alpha^{(21)}, \quad \cos \theta_N^{(21)} = + \cos \alpha^{(21)}, \quad \cos \theta_N^{(21)} = - \cos \theta_i^{(21)}.$$  

Using these relations one can rewrite matrix $A$, Eq. (16), in the following way

$$A_{21} = \begin{pmatrix}
0 & -\cos(\alpha) & 0 & \ldots & 0 & 0 & 0 \\
-\cos(\alpha) & 2\cos(\alpha) & -\cos(\alpha) & \ldots & 0 & 0 & 0 \\
0 & -\cos(\alpha) & 2\cos(\alpha) & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 2\cos(\alpha) & -\cos(\alpha) & 0 \\
0 & 0 & 0 & \ldots & -\cos(\alpha) & 2\cos(\alpha) & -\cos(\alpha) \\
0 & 0 & 0 & \ldots & 0 & -\cos(\alpha) & 2\cos(\alpha)
\end{pmatrix}. \quad (25)$$

Corner minor of $\begin{pmatrix} 2 & 2 \end{pmatrix}$ matrix in the left upper corner of this matrix is equal to

$$d_2 = \begin{vmatrix}
0 & -\cos \alpha \\
-\cos \alpha & 2\cos \alpha
\end{vmatrix} = -\cos^2 \alpha$$

Obviously, it cannot be positive. It follows from this fact that spiral magnetic structure that corresponds to the second branch of the solution and first kind of the solution is always unstable.

### 2.2 Second branch of solution, second kind of solution

$$\alpha^{(22)} = \frac{2 \pi n}{N - 3}, \quad 0 \leq n \leq \frac{N - 3}{2}.$$  

$$\theta_i^{(22)} = \pi - \alpha^{(22)}, \quad \cos \theta_i^{(22)} = - \cos \alpha^{(22)}, \quad \cos \theta_N^{(22)} = - \cos \alpha^{(22)}, \quad \cos \theta_N^{(22)} = + \cos \theta_i^{(22)}$$  

Using these relations one can rewrite matrix $A$, Eq. (16), in the following way

$$A_{22} = \begin{pmatrix}
0 & -\cos(\alpha) & 0 & \ldots & 0 & 0 & 0 \\
-\cos(\alpha) & 2\cos(\alpha) & -\cos(\alpha) & \ldots & 0 & 0 & 0 \\
0 & -\cos(\alpha) & 2\cos(\alpha) & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 2\cos(\alpha) & -\cos(\alpha) & 0 \\
0 & 0 & 0 & \ldots & -\cos(\alpha) & 2\cos(\alpha) & -\cos(\alpha) \\
0 & 0 & 0 & \ldots & 0 & -\cos(\alpha) & 0
\end{pmatrix}. \quad (26)$$

Corner minor of $\begin{pmatrix} 2 & 2 \end{pmatrix}$ matrix in the left upper corner and in the right bottom corner is equal to
\[ d_2 = \begin{vmatrix} 0 & -\cos \alpha \\ -\cos \alpha & 2 \cos \alpha \end{vmatrix} \equiv \begin{vmatrix} 2 \cos \alpha & -\cos \alpha \\ -\cos \alpha & 0 \end{vmatrix} = -\cos^2 \alpha. \]

Obviously it cannot be positive. It follows from this fact that spiral magnetic structure that corresponds to the second branch of the solution and the second kind of solution is always unstable.
Appendix A. Analysis of the sign of the eigenvalues of the matrix $A_{12}$

Let us calculate eigenvalues of matrix $A_{12}$. Eigenvalues $\lambda_M$ of matrix $M_{12}$ and eigenvalues $\lambda_A$ of matrix $A_{12}$ are related by the following relationship $\lambda_A = (2\cos \alpha) \lambda_M$. To find the eigenvalue $\lambda_M$ of matrix $M_{12}$ one should solve the equation

$$\det(M_{12} - \lambda_M I) = \begin{vmatrix} 1 - \lambda_M & -1/2 & 0 & \ldots & 0 & 0 & 0 \\ -1/2 & 1 - \lambda_M & -1/2 & \ldots & 0 & 0 & 0 \\ 0 & -1/2 & 1 - \lambda_M & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 - \lambda_M & -1/2 & 0 \\ 0 & 0 & 0 & \ldots & -1/2 & 1 - \lambda_M & -1/2 \\ 0 & 0 & 0 & \ldots & 0 & -1/2 & -\lambda_M \end{vmatrix} = 0.$$  \hfill (A1)

This matrix differs from matrix (19) because the lower down corner matrix element does not contain a unit. Let us expand this determinant along the last row. Here we introduce the designation $d_n$ which is the determinant of $(n \times n)$ matrix, Eq. (19), where all diagonal elements are the same. Then equation (A1) for eigenvalues of matrix $M_{12}$ can be written as

$$-\lambda d_{N-1} - \frac{1}{4} d_{N-2} = 0 \quad \rightarrow \quad d_{N-1} = d_N.$$  \hfill (A2)

The formula for determinant $d_n$ depends on the interval where we search for the eigenvalue. If we search for them in the interval $0 < \lambda < 2$, then the following inequality $-1 < 1 - \lambda < 1$ is valid. Then one can use the following parametrization: $1 - \lambda = \cos \varphi$, $0 < \varphi < \pi$, i.e. instead of searching for the value of parameter $\lambda$, we shall search for the value of another parameter $\varphi$. In this particular case the formula for $d_n$ is as follows

$$d_n = \frac{\sin \varphi (n+1)}{2^n \sin \varphi}.$$  \hfill (A3)

The substitution of this expression in equation (A2) allows one to rewrite it in a following way

$$f(\varphi) = \frac{\sin \varphi (N+1)}{\sin \varphi N} = 2.$$  \hfill (A4)

The simplest analysis of the function $f(\varphi)$ that figures in the left-hand side of equation (A4), shows that in the interval $0 < \varphi < \pi$ it has $N$ zeros at the points $\varphi_0 = \pi k/(N+1)$, $k = 1, 2, \ldots, N$ and $(N-1)$ vertical asymptotes at the points $\varphi_\alpha = \pi k/N$, $k = 1, 2, \ldots, N-1$. At the point $\varphi = 0$
function \( f(\varphi) \) is undefined but it has finite limit at this point: \( \lim_{\varphi \to 0} f(\varphi) = (N+1)/N \). At the point \( \varphi = \pi \) function \( f(\varphi) \) is undefined too but it has finite limit at this point too: \( \lim_{\varphi \to 0} f(\varphi) = -(N+1)/N \).

It follows from these results that the equation \( f(\varphi) = \gamma \) has \( N \) roots if the following double inequality is satisfied \( -(N+1)/N \leq \gamma \leq (N+1)/N \) and it has \((N-1)\) roots if the inequality \( \gamma > (N+1)/N \) or inequality \( \gamma < -(N+1)/N \) is satisfied. In our equation (A4) the parameter is \( \gamma = 2 \), and thus the equation (A4) for parameter \( \varphi \) has \((N-1)\) roots in the interval \((0, \pi)\). Due to relation \( 1 - \lambda = \cos \varphi \) there is a one-to-one correspondence between magnitudes of \( \varphi \) in the interval \( 0 < \varphi < \pi \) and magnitudes of eigenvalues \( \lambda \) in the interval \( 0 < \lambda < 2 \). Therefore, equation (A1) has exactly \((N-1)\) roots in the interval \( 0 < \lambda < 2 \). However, it is known that the \( M_{12} \) matrix and therefore the \( A_{12} \) matrix must have exactly \( N \) eigenvalues.

![Fig. 1. The plot of function \( f(\varphi) \), defined by the formula (A4).](image-url)

Now let us search for the missing eigenvalue of the matrix \( M_{12} \) in the area of negative values. In order to do this we shall use another kind of parametrization: \( 1 - \lambda = c \varphi \), \( \varphi > 0 \). In this particular case the formula for \( d_n \) can be written as

\[
d_n = \frac{\sinh \varphi (n+1)}{2^n \sinh \varphi}.
\]

(A5)

The substitution of this expression for \( d_n \), Eq. (A5), in equation (A2) for parameter \( \varphi \) allows one to write the equation in the following form...
In the interval \((0, +\infty)\), the function \(f(\varphi)\) monotonically increases from the value \((N+1)/N\) up to \(+\infty\). A function \(f(\varphi)\) is even, \([f(\varphi) = f(-\varphi)]\) that is, in the interval \((-\infty, 0)\), it monotonically decreases. At zero point the function is undefined. We define function \(f(\varphi)\) at zero point by the magnitude of its limit: \[ f(0) \equiv \lim_{\varphi \to 0} f(\varphi) = \frac{(N+1)}{N}. \] Therefore, \(f(\varphi)\) achieves minimal value at zero point and approaches infinity if \(\varphi \to +\infty\). It means that equation \(f(\varphi) = 2\) necessarily has exactly one root in the interval \((0, +\infty)\).

Now one should remind that actually we are interested in eigenvalues \(\lambda_A\) of \(A_{12}\) matrix rather than in the eigenvalues \(\lambda_M\) of \(M_{12}\). They relate to eigenvalues of \(M_{12}\) matrix by the equation \(\lambda_A = (2 \cos \alpha) \lambda_M\). If \(\cos \alpha > 0\), then matrix \(A_{12}\) has one negative eigenvalue and \((N-1)\) positive eigenvalues. If \(\cos \alpha < 0\), then the \(A_{12}\) matrix has \((N-1)\) negative eigenvalues and one positive eigenvalue. Therefore, regardless to the sign of \(\cos \alpha\), the matrix \(A_{12}\) always has negative eigenvalue. The final conclusion is the following: the spiral magnetic structure related to the second kind of solution of the first branch is always unstable.

![Fig. 2. Plot of the function \(f(\varphi)\), defined by the formula (A6).](image-url)
Appendix B. The analysis of the sign of the eigenvalues of the matrix $A_{22}$

The goal of this section is to prove that the matrix $A_{22}$, Eq. (26), has negative eigenvalues. Here we solve a slightly more general problem. Namely, we introduce a parameter named surface perturbation and find its threshold magnitude which has the following sense. If surface perturbation exceeds this threshold magnitude then the minimal eigenvalue of matrix $A_{22}$ becomes negative, else all eigenvalues are positive. In order to do this we rewrite zero matrix element in the upper left and lower right corner of matrix $A_{22}$ in a form $0 = 2\cos\alpha - \Gamma$. $\Gamma$ is surface perturbation introduced. Below we consider the general case when the magnitude of $\Gamma$ is assumed to be arbitrary. However in the particular case of matrix $A_{22}$ determined by Eq. (26) one should put $\Gamma = 2\cos\alpha$. Then the modernized matrix coincides with matrix $A_{22}$, Eq. (26).

Equation for eigenvalues $\lambda_M$ of the reduced matrix $M_{22} = A_{22}/(2\cos\alpha)$ can be written as

$$
\det \left( M_{22} - \lambda_M I \right) = 0
$$

$$
\begin{pmatrix}
1 - \gamma - \lambda_M & -1/2 & 0 & \ldots & 0 & 0 & 0 \\
-1/2 & 1 - \lambda_M & -1/2 & \ldots & 0 & 0 & 0 \\
0 & -1/2 & 1 - \lambda_M & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 - \lambda_M & -1/2 & 0 \\
0 & 0 & 0 & \ldots & -1/2 & 1 - \lambda_M & -1/2 \\
0 & 0 & 0 & \ldots & 0 & -1/2 & 1 - \gamma - \lambda_M
\end{pmatrix}
$$

(B1)

Eigenvalues $\lambda_M$ of matrix $M_{22}$ and eigenvalues $\lambda_A$ of matrix $A_{22}$ are related by the equality $\lambda_M = (2\cos\alpha)\lambda_A$. This goes for the surface perturbations $\Gamma = (2\cos\alpha)\gamma$.

Let us designate as $d_n$ the determinant of three-diagonal $(n \times n)$ matrix with equal diagonal elements $\varepsilon$

$$
d_n = \det \begin{pmatrix}
\varepsilon & -1/2 & 0 & 0 & \ldots & 0 & 0 \\
-1/2 & \varepsilon & -1/2 & 0 & 0 & \ldots & 0 \\
0 & -1/2 & \varepsilon & \ldots & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \varepsilon & -1/2 & 0 \\
0 & \ldots & 0 & \varepsilon & -1/2 & \varepsilon & -1/2 \\
0 & 0 & \ldots & 0 & -1/2 & \varepsilon & -1/2
\end{pmatrix}
$$

(B2)

Compact formula for $d_n$ depends on the interval the parameter $\varepsilon$ belongs to.
\[
d_{n} = \begin{cases} 
\frac{\sin \varphi(n+1)}{2^n \sin \varphi}, & -1 \leq \varepsilon \leq +1, \quad \varepsilon = \cos \varphi, \quad 0 \leq \varphi \leq \pi \\
\frac{\sinh \varphi(n+1)}{2^n \sinh \varphi}, & 1 < \varepsilon, \quad \varepsilon = \cosh \varphi, \quad 0 < \varphi \\
(-1)^{n} \frac{\sinh \varphi(n+1)}{2^n \sinh \varphi}, & \varepsilon < -1, \quad \varepsilon = -\cosh \varphi, \quad 0 < \varphi 
\end{cases}
\] (B3)

Determinant (B1) can be expressed via determinants \(d_{n}\) of matrix (B2).

\[
d_{N} - 2\gamma d_{N-1} + \gamma^2 d_{N-2} = 0.
\] (B4)

Consider each interval for \(\varepsilon\) separately: \(|\varepsilon| < 1\), \(\varepsilon > 1\) and \(\varepsilon < -1\). Each of these intervals for \(\varepsilon\) corresponds to definite interval for the eigenvalue \(\lambda\).

1. \(|\varepsilon| < 1 \rightarrow -1 < \lambda < +1 \rightarrow 0 < \lambda < 2\). Then substituting into (B4) the expression for \(d_{n}\) from the first line in (B3) one can rewrite (B4) in a following form

\[
4\gamma^2 \sin \varphi(N-1) - 4\gamma \sin \varphi N + \sin \varphi(N+1) = 0.
\] (B5)

Solution of this quadratic equation with respect to parameter \(\gamma\) gives

\[
\gamma_{\pm} = \frac{\sin \varphi N + \sin \varphi}{2 \sin \varphi(N-1)} = \frac{\sin \varphi(N+1)/2}{2 \sin \varphi(N-1)/2} \equiv g_{\pm}(\varphi)
\]

Elementary analysis of functions \(g_{\pm}(\varphi)\) shows the following. Total number of zeros of these functions in the interval \((0, \pi)\) is equal to \(N\). The upper equation in (B6) loses its minimal positive root if \(\gamma\) exceeds the threshold magnitude \((N+1)/2(N-1)\). The lower equation in (B6) loses its minimal positive root if \(\gamma\) exceeds the threshold magnitude \(1/2\). In our case surface perturbation \(\gamma = 1\), because of that inside the “band” \(0 < \lambda < 4\) set of equations has \((N-2)\) roots. Let us prove that in the particular case \(\gamma = 1\) these two missed roots appear in the interval \(-\infty < \lambda < 0\). To prove this statement consider the case \(\varepsilon > 1\).

2. \(\varepsilon > 1 \rightarrow 1 < 1 - \lambda \rightarrow \lambda < 0\). Then substituting in (B4) the expression for \(d_{n}\) from the second line in (B3) we shall get

\[
4\gamma^2 \sinh \varphi(N-1) - 4\gamma \sinh \varphi N + \sinh \varphi(N+1) = 0.
\] (B7)

Solution of this quadratic equation with respect to parameter \(\gamma\) gives
Eqs. (B8) can be considered as definitions of functions $f_\pm$ rather than only equations for parameter $\varphi$. Consider, first, function $f_+ (\varphi)$ in (B8). It is an odd function: $f_+ (\varphi) = f_+ (-\varphi)$.

Let us prove that for positive magnitudes $\varphi$ function $f_+ (\varphi)$ monotonically increases. To get convinced of this statement, let us calculate the derivative of this function and investigate its sign.

$$\frac{df_+ (\varphi)}{d\varphi} = \frac{\sinh \varphi N - N \sinh \varphi}{2 \cosh (N-1)}.$$  \hspace{1cm} (B9)

Obviously the denominator of this fraction is positive. Let us prove that the numerator is positive for positive values of $\varphi$.

$$\sinh \varphi N - N \sinh \varphi > 0.$$  \hspace{1cm} (B10)

To prove this statement for any magnitude of integer parameter $N$ we use the method of mathematical induction. For $N = 2$ the statement (B10) is true because the left part of this inequality can be rewritten as $2 \sinh \varphi (\cosh \varphi - 1)$, which is obviously positive. Now let us assume that statement (B10) is true for some value $N$ and prove that it is also true for $(N+1)$, i.e. the following inequality is satisfied.

$$\sinh \varphi (N+1) - (N+1) \sinh \varphi > 0.$$  \hspace{1cm} (B11)

We rewrite inequality (B11) in a following form

$$\sinh \varphi (N+1) - N \sinh \varphi - \sinh \varphi > 0.$$  \hspace{1cm} (B12)

We rewrite inequality (B10) in a following form

$$-N \sinh \varphi > -\sinh \varphi N.$$  \hspace{1cm} (B13)

And add term $\sinh \varphi (N+1) - \sinh \varphi$ to both sides in (B13). The result is the following.

$$\sinh \varphi (N+1) - N \sinh \varphi - \sinh \varphi > \sinh \varphi (N+1) - \sinh \varphi - \sinh \varphi N.$$  \hspace{1cm} (B14)

Now we transform the right part in (B14) in the following way
\[
\sinh \varphi (N + 1) - \left[ \sinh \varphi + \sinh \varphi N \right] = \\
= 2 \sinh \frac{\varphi (N + 1)}{2} \cosh \frac{\varphi (N + 1)}{2} - 2 \sinh \frac{\varphi (N + 1)}{2} \cosh \frac{\varphi (N - 1)}{2} = \\
= 2 \sinh \frac{\varphi (N + 1)}{2} \left[ \cosh \frac{\varphi (N + 1)}{2} - \cosh \frac{\varphi (N - 1)}{2} \right]
\]

(B 15)

Obviously the final expression is positive for \( \varphi > 0 \). This in turn means that the left side of inequality (B14) is positive. Hence the inequality (B11) is true. This is exactly what we wanted to prove. Therefore, in Eq. (B8) function \( f_+ (\varphi) \) monotonically increases from the value \( f_+ (0) = (N + 1)/2(N - 1) \) up to infinity. If \( \varphi \to +\infty \), then the function approaches \( +\infty \) as \( \exp(\varphi)/2 \). Since in our particular case \( \gamma = 1 \), the upper equation in (B8) has one root \( \varphi \) in the interval \( (0, +\infty) \). This in turn means the existence of a negative eigenvalue \( \lambda \) corresponding to this root.

Now let us investigate the function \( f_- (\varphi) \) in (B8). This is an odd function: \( f_- (\varphi) = f_+ (-\varphi) \). Let us prove that for positive values \( \varphi \), the function \( f_- (\varphi) \) monotonically increases. In order to do this one should calculate the derivative of this function and investigate its sign.

\[
\frac{df_- (\varphi)}{d\varphi} = \frac{\sinh \varphi N + N \sinh \varphi}{2 \left[ \cosh \varphi (N - 1) + 1 \right]}
\]

(B16)

Obviously this derivative is positive for \( \varphi > 0 \). Therefore, the function \( f_- (\varphi) \) in (B8) monotonically increases from the value \( f_- (0) = 1 \) to infinity. If \( \varphi \to +\infty \) then the function \( f_- (\varphi) \) approaches \( +\infty \) as \( \exp(\varphi)/2 \). Since in our particular case \( \gamma = 1 \), the lower equation in (B8) has one root \( \varphi \) in the interval \( (0, +\infty) \). This in turn means the existence of a negative eigenvalue \( \lambda \) corresponding to this root.

3. \( \varepsilon < -1 \to -\lambda < -1 \to 2 < \lambda \). Then substituting the expression for \( d_n \) from the third line in (B3) in equation (B4) we get

\[
4\gamma^2 \sinh \varphi (N - 1) + 4\gamma \sinh \varphi N + \sinh \varphi (N + 1) = 0.
\]

(B17)

Solution of this quadratic equation with respect to \( \gamma \) gives

\[
\begin{bmatrix}
\gamma_+ = -\frac{\sinh \varphi N + \sinh \varphi}{2 \sinh \varphi (N - 1)} = -\frac{\cosh \varphi (N + 1)/2}{2 \cosh \varphi (N - 1)/2} = -f_- (\varphi) \\
\gamma_- = -\frac{\sinh \varphi N - \sinh \varphi}{2 \sinh \varphi (N - 1)} = -\frac{\sinh \varphi (N + 1)/2}{2 \sinh \varphi (N - 1)/2} = -f_+ (\varphi)
\end{bmatrix}
\]

(B18)

Note that functions \( f_+ (\varphi) \) and \( f_- (\varphi) \) are defined by formulas (B8). In the interval \( (0, +\infty) \) the
function \(-f_-(\varphi)\) monotonically decreases from \(-f_-(0) = -1\) down to \(-\infty\), and the function \(-f_+(\varphi)\) monotonically decreases from \(-f_+(0) = (N+1)/2(N-1)\) down to \(-\infty\). By this reason for \(\gamma = 1\) none of the two equations (B18) have roots \(\varphi\) in this interval. This in turn means the absence of eigenvalues \(\lambda > 2\).

Therefore, for \(\gamma = 1\) the matrix \(M_{22}\) in equation (B1) always has two negative eigenvalues. Due to the relation \(\lambda_A = (2\cos \alpha) \lambda_M\), it means that for \(\cos \alpha > 0\) the matrix \(A_{22}\), Eq. (26), has two negative eigenvalues. If however \(\cos \alpha < 0\) then due to the same relation \(\lambda_A = (2\cos \alpha) \lambda_M\), the matrix \(A_{22}\), Eq. (26), has \((N-2)\) negative eigenvalues. Therefore, the matrix \(A_{22}\) always has negative eigenvalues.