Appendix A

Parameter Estimation of SHFMM utilizing First-Order Baum-Welch Algorithm

The First-Order semi-hidden Fritchman Markov model for a discrete power line communication channel is described by the $3 \times 3$ state transition probability matrix $A_1$ and the $2 \times 3$ input-to-output symbol transition probabilities otherwise referred to as the error probability generation matrix $B$. An iterative process for estimation of the First-Order SHFMM full parameters $\Gamma_1 = \{A_1, B\}$ from a measured bit error sequence, $\bar{E} = [e_1, \ldots, e_t, \cdots e_T]$ is based on the First-Order Baum-Welch algorithm (BWA). This iterative BWA by design converges to the maximum likelihood estimator $\Gamma_1 = \{A_1, B\}$ that maximizes $\Pr(\bar{E}|\Gamma_1)$. The objective is computing the estimates of the element of the First-Order state transition probability matrix $A_1$. For a given three-state SHFMM, with two good states and a single bad state. Transition is not allowed between the two good states, hence, $A_1$ is mathematically shown in Equation A.1 as follows.

$$A_1 = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ (A.1)

Based on the restriction in transition, the error probability generation matrix $B$ is written in binary form as follows.

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (A.2)

And lastly, the initial state or prior probability $\Pi$, represents the probability of being in any of the three states at a particular time. The $\Pi$ vector for the adopted three state SHFMM is written as follow. Note $\sum_{i=1}^{N} = 1$, for $N = 3$ (number of model states).

$$\Pi = [\pi_1 \pi_2 \cdots \pi_N]$$ (A.3)

$$\Pi = [\pi_1 \pi_2 \pi_3]$$ (A.4)
The computations needed to implement the First-Order Baum-Welch algorithm are described as follows.

**A.1 Step 1: Model Initialization**

Initialize the adopted model $\Gamma_1 = \{A_1, B\}$ and $\Pi$, the initial state probability vector.

**A.2 Step 2: Computation of Forward and Backward Probability Variables**

Given $\Gamma_1 = \{A_1, B\}$ as the adopted model, $\Pi$, the initial state probability vector, and measured bit error sequence, $\bar{E} = [e_1, e_2, \cdots, e_T]$, where $T$ is the length of the bit error sequence. We first compute the “forward probability variables”

$$\alpha_t(i) = Pr[e_1, e_2, \cdots, e_t, s_t = S_i | \Gamma_1] \quad (A.5)$$

$$\alpha_t(i) = \pi_i b_i (e_t) \quad (A.6)$$

As well as the “backward probability variables”

$$\beta_t = Pr[e_{t+1}, e_{t+2}, \cdots, e_T | s_t = S_i, \Gamma_1] \quad (A.7)$$

For $t = 1, 2, \cdots, T$ and $i = 1, 2, \cdots, N$. Details of the computation of the forward and backward probability variables are presented as follows.

**A.2.1 Forward Probability Variables Computation**

The computation of the forward probability variables is executed in three steps: initialization, induction and termination procedure.

**A.2.1.1 Initialization procedure :**

$$\alpha_1(i) = \pi_i b_i (e_1), i = 1, 2, \cdots, N \quad (A.8)$$
A.2.1.2 Induction procedure:

\[ \alpha_{t+1}(j) = \left[ \sum_{i=1}^{N} \alpha_t(i)a_{ij} \right] b_j(e_{t+1}), \ 1 \leq t \leq T - 1, 1 \leq j \leq N \]  
(A.9)

A.2.1.3 Termination procedure:

\[ \Pr[\bar{E}|\Gamma_1] = \sum_{i=1}^{N} \alpha_T(i)\beta_T(i) \]  
(A.10)

Note that, \( \beta_T(i) = 1 \), for \( i = 1, 2, \cdots, N \). Therefore,

\[ \sum_{i=1}^{N} \alpha_T(i) = \sum_{i=1}^{N} \Pr[e_1, e_2, \cdots, e_T, s_T = S_i|\Gamma_1] = \Pr[\bar{E}|\Gamma_1] \]  
(A.11)

This means,

\[ \Pr[\bar{E}|\Gamma_1] = \sum_{i=1}^{N} \alpha_T(i) \]  
(A.12)

\[ \Pr[\bar{E}|\Gamma_1] = \sum_{i=1}^{N} \pi_i b_i(e_T) \]  
(A.13)

A.2.2 Backward Probability Variables Computation

The computation of the backward probability variables unlike the forward probability variables is executed in two steps: the initialization and induction steps as follows.

A.2.2.1 Initialization procedure:

\[ \beta_T(i) = 1, \ i = 1, 2, \cdots, N \]  
(A.14)

A.2.2.2 Induction procedure:

\[ \beta_t(i) = \sum_{j=1}^{N} \beta_{t+1}(j)b_j(e_{t+1})a_{ij}, \ 1 \leq t \leq T - 1, 1 \leq j \leq N \]  
(A.15)
A.3 Step 3: Computation of Model Parameter Re-estimation Variables

The first re-estimation variable to be computated is $\gamma_t(i)$. $\gamma_t(i)$ denotes the expected number of transitions from $i$, which is the probability of being in state $i$ at time $t$, given the model $\Gamma_1$ and the measured bit error sequence $\bar{E}$. Computation of $\gamma_t(i)$ is mathematically shown as follows.

$$\gamma_t(i) = \Pr[s_t = S_i | \bar{E}, \Gamma_1] = \frac{\alpha_t(i) \beta_t(i)}{\Pr[E | \Gamma_1]}, \quad i = 1, 2, \ldots, N$$ (A.16)

$$\gamma_t(i) = \frac{\alpha_t(i) \beta_t(i)}{\sum_{i=1}^{N} \alpha_t(i) \beta_t(i)}, \quad i = 1, 2, \ldots, N$$ (A.17)

The computation of the second re-estimation variable $\xi_t(i,j)$ is also mathematically shown as follows. $\xi_t(i,j)$ denotes the expected number of transitions from state $i$ to state $j$, which is the probability of being in state $i$ at time $t$, and being in state $j$ at time $t + 1$, given the model $\Gamma_1$ and the measured bit error sequence $\bar{E}$.

$$\xi_t(i,j) = \Pr[s_t = S_i, s_{t+1} = S_j | \bar{E}, \Gamma_1]$$ (A.18)

$$\xi_t(i,j) = \frac{\alpha_t(i) a_{ij} b_{j(e_{t+1})} \beta_{t+1}(j)}{\Pr[E | \Gamma_1]}$$ (A.19)

$$\xi_t(i,j) = \frac{\alpha_t(i) a_{ij} b_{j(e_{t+1})} \beta_{t+1}(j)}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_t(i) a_{ij} b_{j(e_{t+1})} \beta_{t+1}(j)}$$ (A.20)

A.4 Step 4: Computation of the Model Parameter Estimates

We now compute the new First-Order state transition probability elements $\hat{a}_{ij}$ using the expected frequencies derived in the previous step.

$$\hat{a}_{ij} = \frac{\text{expected number of transitions from } i \text{ to } j}{\text{expected number of transitions from } i}$$ (A.21)

$$\hat{a}_{ij} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)}$$ (A.22)
Appendix A. First-Order BWA for SHFMM Parameter Estimation

Note, since the input-to-output symbol transition otherwise referred to as the error probability generation matrix $B$ is in binary form due to restriction in transition for the adopted three-state SHFMM, the estimates of the $B$ is not computed.

The estimate of the initial state probability vector, $\hat{\pi}_i$, which is the expected number of times in state $S_i$ at time $t = 1$ is computed as follows.

$$\hat{\pi}_i = \alpha_t(i)\beta_t(i), \ i = 1, 2, \cdots, N \quad (A.23)$$

A.5 Step 5:

Return to Step 2 with the new parameter estimates $\hat{\Gamma}_1 = \{A_1, B, \hat{\pi}\}$, or equivalently $\hat{\Gamma}_1 = \Gamma_1$, obtained in Step 4 and replicate the process till the desired convergence is attained.
Appendix B

Parameter Estimation of SHFMM utilizing Second-Order
Baum-Welch Algorithm

The Second-Order semi-hidden Fritchman Markov model for a discrete power line communication channel is described by the $9 \times 3$ state transition probability matrix $A_2$ and the $2 \times 3$ input-to-output symbol transition probabilities otherwise referred to as the error probability generation matrix $B$. An iterative process for estimation of the Second-Order SHFMM full parameters $\Gamma_2 = \{A_2, B\}$ from a measured bit error sequence, $\bar{E} = [e_1, \ldots, e_t, \ldots e_T]$ is based on the Second-Order Baum-Welch algorithm (BWA). This iterative BWA by design converges to the maximum likelihood estimator $\Gamma_2 = \{A_2, B\}$ that maximizes $\Pr(\bar{E}|\Gamma_2)$. The objective is computing the estimates of the element of the Second-Order state transition probability matrix $A_2$. For a given three-state SHFMM, with two good states and a single bad state. Transition is not allowed between the two good states, hence, $A_1$ and $A_2$ is mathematically shown in Equation B.1 and Equation B.2 respectively.

$$A_1 = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$  \hspace{1cm} \text{(B.1)}$$

$$A_2 = \begin{bmatrix} a_{111} & 0 & a_{113} \\ 0 & a_{122} & a_{123} \\ a_{131} & a_{132} & a_{133} \\ a_{211} & 0 & a_{213} \\ 0 & a_{222} & a_{223} \\ a_{231} & a_{232} & a_{233} \\ a_{311} & 0 & a_{313} \\ 0 & a_{322} & a_{323} \\ a_{331} & a_{332} & a_{333} \end{bmatrix}$$  \hspace{1cm} \text{(B.2)}$$
Based on the restriction in transition, the error probability generation matrix $B$ is written in binary form as follows.

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (B.3)$$

And lastly, the initial state or prior probability $\Pi$, represents the probability of being in any of the three states at a particular time. The $\Pi$ vector for the adopted three state Second-Order SHFMM is written as follow. Note $\sum_{i=1}^{N} \pi_i = 1$, where $N$ is the number states in the SHFMM).

$$\Pi = [\pi_1 \, \pi_2 \, \cdots \, \pi_N] \quad (B.4)$$

$$\Pi = [\pi_1 \, \pi_2 \, \pi_3], \quad for \ N = 3 \quad (B.5)$$

The computations needed to implement the Second-Order Baum-Welch algorithm are described as follows.

### B.1 Step 1: Model Initialization

Initialize the adopted model $\Gamma_2 = \{A_2, B\}$ and $\Pi$, the initial state probability vector.

### B.2 Step 2: Computation of Forward and Backward Probability Variables

Given $\Gamma_2 = \{A_2, B\}$ as the adopted model, $\Pi$, the initial state probability vector, and measured bit error sequence, $\bar{E} = [e_1, e_2, \cdots, e_T]$, where $T$ is the length of the bit error sequence. We first compute the “forward probability variables” $\alpha_t(i,j)$.

The forward probability variables denoted by $\alpha_t(i,j)$, is the probability of the partial bit error sequence from 1 to time $t$, and transition $S_i \rightarrow S_j$ at times $t-1$, $t$ given the model $\Gamma_2=(A_2, B)$. $\alpha_t(i,j)$ is written in mathematical form as follows.

$$\alpha_t(i,j) = Pr(e_1, e_2, \cdots, e_t, \ s_{t-1} = S_i, s_t = S_j | \Gamma_2) \quad (B.6)$$
The “backward probability variables” is also computed. The \textit{backward probability variables} denoted by $\beta_t(i,j)$, is defined as the probability of the partial bit error sequence $t+1$ to $T$, given the transition $S_i, S_j$ at times $t-1, t$ and the model $\Gamma_2={A_2, B}$. $\beta_t(i,j)$, is written in mathematical form as follows.

\begin{equation}
\beta_t(i,j) = \Pr(e_{t+1}, e_{t+2}, e_{t+3}, \ldots, e_T|s_{t-1} = S_i, s_t = S_j, \Gamma_2)
\end{equation}

\subsection{B.2.1 Forward Probability Variables Computation}

The computation of the forward probability variables is carried out in three steps: initialization, induction and termination procedure. First, we compute

\begin{equation}
\alpha_1(i) = \pi_i b_i (e_1), \ i = 1, 2, \ldots, N
\end{equation}

Then, other procedures are carried out as follows.

\subsection*{B.2.1.1 Initialization procedure:}

\begin{equation}
\alpha_2(i,j) = \alpha_1(i) a_{ij} b_j (e_2), \ for \ 1 \leq i, j \leq N
\end{equation}

\subsection*{B.2.1.2 Induction procedure:}

\begin{equation}
\alpha_{t+1}(j,k) = \left[ \sum_{i=1}^{N} \alpha_t(i,j) a_{ijk} \right] b_k (e_{t+1}), \ for \ 2 \leq t \leq T - 1
\end{equation}

\subsection*{B.2.1.3 Termination procedure:}

\begin{equation}
\Pr(\bar{E}|\Gamma_2) = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_T(i,j)
\end{equation}

\subsection{B.2.2 Backward Probability Variables Computation}

The computation of the backward probability variables unlike the forward probability variables is executed in two steps: the initialization and induction steps as follows.
B.2.2.1 Initialization procedure:

\[ \beta_t(i,j) = 1, \text{ for } 1 \leq i,j \leq N \] (B.12)

B.2.2.2 Recursive procedure:

\[ \beta_t(i,j) = \sum_{k=1}^{N} a_{ijk} b_k(e_{t+1}) \beta_{t+1}(j,k), \text{ for } T - 1 \geq t \geq 2 \] (B.13)

Computation of both forward and backward functions requires in the order of \( N^3T \) calculations.

B.3 Step 3: Computation of Model Parameter Re-estimation Variables

The first parameter re-estimation variable to be computed is denoted by \( \eta_t(i,j,k) \) and defined as the probability of being in states \( S_i, S_j \) and \( S_k \) respectively at times \( t-1, t \) and \( t+1 \) given the Second-Order model \( \Gamma_2 = \{A_2, B\} \) and measured bit error sequence \( \bar{E} \). \( \eta_t(i,j,k) \) is computed as follows.

\[ \eta_t(i,j,k) = \Pr(s_{t-1} = S_i, s_t = S_j, s_{t+1} = S_k | \bar{E}, \Gamma_2) \] (B.14)

\[ \eta_t(i,j,k) = \frac{\alpha_t(i,j) a_{ijk} b_k(e_{t+1}) \beta_{t+1}(j,k)}{\Pr(\bar{E} | \Gamma_2)} \] (B.15)

\[ \eta_{t+1}(i,j,k) = \frac{\alpha_{t+1}(i,j) a_{ijk} b_k(e_{t+2}) \beta_{t+2}(j,k)}{\Pr(\bar{E} | \Gamma_2)} \] (B.16)

The second parameter re-estimation variable to be computed is denoted by \( \xi_t(i,j) \) and defined as the probability of being in state \( S_i \) at time \( t \) and in state \( S_j \) at time \( t+1 \) given the Second-Order model \( \Gamma_2 = \{A_2, B\} \) and measured bit error sequence \( \bar{E} \). \( \xi_t(i,j) \) is computed as follows.

\[ \xi_t(i,j) = \Pr(s_t = S_i, s_{t+1} = S_j | \bar{E}, \Gamma_2) \] (B.17)
Appendix B. Second-Order BWA for SHFMM Parameter Estimation

\[ \xi_t(i, j) = \sum_{k=1}^{N} \eta_{t+1}(i, j, k) \]  
\[ \xi_t(i, j) = \sum_{k=1}^{N} \frac{\alpha_{t+1}(i, j) a_{ijk} b_k (e_{t+2}) \beta_{t+2}(j, k)}{\Pr(\bar{E}|\Gamma_2)} \]  

The last parameter re-estimation variable to be computed is denoted by \( \gamma_t(i) \) and defined as the probability of being in state \( S_i \) at time \( t \), given the Second-Order model \( \Gamma_2 = \{ A_2, B \} \) and measured bit error sequence \( \bar{E} \). \( \gamma_t(i) \) is computed as follows.

\[ \gamma_t(i) = \Pr(s_t = S_i|\bar{E}, \Gamma_2) \]  
\[ \gamma_t(i) = \sum_{j=1}^{N} \xi_t(i, j) \]  
\[ \gamma_t(i) = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\alpha_{t+1}(i, j) a_{ijk} b_k (e_{t+2}) \beta_{t+2}(j, k)}{\Pr(\bar{E}|\Gamma_2)} \]  

B.4 Step 4: Computation of the Model Parameter Estimates

We now compute the new First-Order state transition probability elements \( \hat{a}_{ij} \) using the parameter re-estimation variables computed in the previous step as follows.

\[ \hat{a}_{ij} = \frac{\xi_1(i, j)}{\gamma_1(i)} \]

We also compute the new Second-Order state transition probability elements \( \hat{a}_{ijk} \) as follows.

\[ \hat{a}_{ijk} = \frac{\sum_{t=1}^{T-3} \eta_{t+1}(i, j, k)}{\sum_{t=1}^{T-3} \xi_t(i, j)} \]

Note, since the input-to-output symbol transition otherwise referred to as the error probability generation matrix \( B \) is in binary form due to restriction in transition for the adopted three-state SHFMM, the estimates of the \( B \) is not computed.
Appendix B. Second-Order BWA for SHFMM Parameter Estimation

The estimate of the initial state probability vector, $\hat{\pi}_i$, is computed as follows.

$$\hat{\pi}_i = \frac{\gamma_1(i)}{\sum_{i=1}^{N} \gamma_1(i)}$$ (B.25)

B.5 Step 5:

Return to Step 2 with the new parameter estimates $\hat{\Gamma}_2 = \{A_2, B, \hat{\pi}\}$, or equivalently $\hat{\Gamma}_2 = \Gamma_2$, obtained in Step 4 and replicate the process till the desired convergence is attained.

A summary of the procedural steps in carrying out model parameter re-estimation using the extended Second-Order Baum-Welch algorithm is as follows.

1. The initialization of $\pi_i^0$, $a_{ij}^0$, $a_{ijk}^0$ and $b_k^0(l)$, for $1 \leq i, j, k \leq N, 1 \leq l \leq M$.

2. Computation of the forward and backward probabilities.

3. Computation of the re-estimation formulas: $\eta_t(i, j, k)$, $\xi_t(i, j)$ and $\gamma_t(i)$, for $1 \leq i, j, k \leq N, 2 \leq t \leq T - 1$ using the computed forward and backward probabilities.

4. Computation of the new re-estimated parameters: $\hat{\pi}_i$, $\hat{a}_{ij}$, $\hat{a}_{ijk}$ and $\hat{b}_k(l)$ for $1 \leq i, j, k \leq N, 1 \leq l \leq M$ utilizing the parameter re-estimation formulas.

5. Reiteration of steps 2-4 with the new re-estimated model parameters until the desired level of convergence is reached, that is, $\pi_i = \hat{\pi}_i$, $a_{ij} = \hat{a}_{ij}$, $a_{ijk} = \hat{a}_{ijk}$ and $b_k(l) = \hat{b}_k(l)$ for $1 \leq i, j, k \leq N, 1 \leq l \leq M$. 
Appendix C

Initial State Transition Probabilities for the First-Order SHFMMs

**Table C.1:** First-Order SHFMM initial state transition probabilities (model 1 - 20)

<table>
<thead>
<tr>
<th></th>
<th>$A_1^1$</th>
<th>$A_1^2$</th>
<th>$A_1^3$</th>
<th>$A_1^4$</th>
<th>$A_1^5$</th>
<th>$A_1^6$</th>
<th>$A_1^7$</th>
<th>$A_1^8$</th>
<th>$A_1^9$</th>
<th>$A_1^{10}$</th>
<th>$A_1^{11}$</th>
<th>$A_1^{12}$</th>
<th>$A_1^{13}$</th>
<th>$A_1^{14}$</th>
<th>$A_1^{15}$</th>
<th>$A_1^{16}$</th>
<th>$A_1^{17}$</th>
<th>$A_1^{18}$</th>
<th>$A_1^{19}$</th>
<th>$A_1^{20}$</th>
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<tr>
<td>$a_{11}$</td>
<td>0.65</td>
<td>0.75</td>
<td>0.85</td>
<td>0.95</td>
<td>0.90</td>
<td>0.80</td>
<td>0.89</td>
<td>0.79</td>
<td>0.95</td>
<td>0.85</td>
<td>0.92</td>
<td>0.90</td>
<td>0.95</td>
<td>0.65</td>
<td>0.85</td>
<td>0.75</td>
<td>0.88</td>
<td>0.98</td>
<td>0.90</td>
<td>0.95</td>
</tr>
<tr>
<td>$a_{13}$</td>
<td>0.35</td>
<td>0.25</td>
<td>0.15</td>
<td>0.05</td>
<td>0.10</td>
<td>0.20</td>
<td>0.11</td>
<td>0.21</td>
<td>0.05</td>
<td>0.15</td>
<td>0.08</td>
<td>0.10</td>
<td>0.05</td>
<td>0.25</td>
<td>0.15</td>
<td>0.25</td>
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<td>0.02</td>
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<td>0.05</td>
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<td>0.85</td>
<td>0.95</td>
<td>0.89</td>
<td>0.95</td>
<td>0.85</td>
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<td>0.85</td>
<td>0.95</td>
<td>0.85</td>
<td>0.88</td>
<td>0.85</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
<td>0.85</td>
</tr>
<tr>
<td>$a_{31}$</td>
<td>0.35</td>
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<td>0.45</td>
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<td>0.40</td>
<td>0.65</td>
<td>0.55</td>
<td>0.50</td>
<td>0.46</td>
<td>0.55</td>
<td>0.45</td>
<td>0.55</td>
<td>0.20</td>
<td>0.37</td>
<td>0.55</td>
<td>0.45</td>
<td>0.55</td>
<td>0.23</td>
<td>0.30</td>
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<tr>
<td>$a_{32}$</td>
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<td>0.45</td>
<td>0.41</td>
<td>0.50</td>
<td>0.40</td>
<td>0.50</td>
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<td>0.32</td>
<td>0.38</td>
<td>0.39</td>
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<td>0.10</td>
<td>0.13</td>
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<td>0.10</td>
<td>0.10</td>
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<td>0.10</td>
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<td>0.10</td>
</tr>
</tbody>
</table>

**Table C.2:** First-Order SHFMM initial state transition probabilities (model 21 - 40)

|       | $A_1^{21}$ | $A_1^{22}$ | $A_2^{22}$ | $A_1^{23}$ | $A_2^{23}$ | $A_1^{24}$ | $A_2^{24}$ | $A_1^{25}$ | $A_2^{25}$ | $A_1^{26}$ | $A_2^{26}$ | $A_1^{27}$ | $A_2^{27}$ | $A_1^{28}$ | $A_2^{28}$ | $A_1^{29}$ | $A_2^{29}$ | $A_1^{30}$ | $A_2^{30}$ | $A_1^{31}$ | $A_2^{31}$ | $A_1^{32}$ | $A_2^{32}$ | $A_1^{33}$ | $A_2^{33}$ | $A_1^{34}$ | $A_2^{34}$ | $A_1^{35}$ | $A_2^{35}$ | $A_1^{36}$ | $A_2^{36}$ | $A_1^{37}$ | $A_2^{37}$ | $A_1^{38}$ | $A_2^{38}$ | $A_1^{39}$ | $A_2^{39}$ | $A_1^{40}$ | $A_2^{40}$ |
|-------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| $a_{11}$ | 0.80      | 0.90      | 0.95      | 0.90      | 0.95      | 0.90      | 0.80      | 0.90      | 0.80      | 0.70      | 0.80      | 0.90      | 0.69      | 0.79      | 0.99      | 0.89      | 0.59      | 0.89      | 0.83      | 0.83      | 0.83      | 0.83      | 0.83      | 0.83      | 0.83      | 0.83      | 0.83      | 0.83      | 0.83      |
| $a_{13}$ | 0.20      | 0.10      | 0.05      | 0.10      | 0.05      | 0.10      | 0.20      | 0.10      | 0.20      | 0.05      | 0.30      | 0.20      | 0.10      | 0.31      | 0.21      | 0.01      | 0.11      | 0.41      | 0.11      | 0.17      | 0.17      | 0.17      | 0.17      | 0.17      | 0.17      | 0.17      | 0.17      | 0.17      | 0.17      |
| $a_{22}$ | 0.90      | 0.92      | 0.85      | 0.80      | 0.90      | 0.85      | 0.72      | 0.82      | 0.92      | 0.62      | 0.90      | 0.92      | 0.85      | 0.82      | 0.92      | 0.82      | 0.92      | 0.95      | 0.93      | 0.93      | 0.93      | 0.93      | 0.93      | 0.93      | 0.93      | 0.93      | 0.93      | 0.93      | 0.93      |
| $a_{31}$ | 0.10      | 0.08      | 0.15      | 0.20      | 0.15      | 0.28      | 0.18      | 0.08      | 0.38      | 0.10      | 0.08      | 0.15      | 0.18      | 0.08      | 0.08      | 0.05      | 0.08      | 0.05      | 0.08      | 0.05      | 0.08      | 0.05      | 0.08      | 0.05      | 0.08      | 0.05      | 0.08      | 0.05      | 0.08      | 0.05      |
| $a_{32}$ | 0.35      | 0.45      | 0.15      | 0.45      | 0.50      | 0.40      | 0.45      | 0.50      | 0.60      | 0.70      | 0.50      | 0.40      | 0.40      | 0.30      | 0.50      | 0.30      | 0.50      | 0.65      | 0.40      | 0.40      | 0.26      | 0.52      | 0.26      | 0.52      | 0.26      | 0.52      | 0.26      | 0.52      | 0.26      | 0.52      |
| $a_{33}$ | 0.50      | 0.45      | 0.75      | 0.48      | 0.50      | 0.42      | 0.50      | 0.43      | 0.40      | 0.35      | 0.20      | 0.45      | 0.51      | 0.52      | 0.65      | 0.40      | 0.60      | 0.40      | 0.26      | 0.52      | 0.26      | 0.52      | 0.26      | 0.52      | 0.26      | 0.52      | 0.26      | 0.52      | 0.26      | 0.52      |
Table C.3: First-Order SHFMM initial state transition probabilities (model 41 - 60)

|   | $A_{11}^1$ | $A_{12}^1$ | $A_{13}^1$ | $A_{14}^1$ | $A_{15}^1$ | $A_{16}^1$ | $A_{17}^1$ | $A_{18}^1$ | $A_{19}^1$ | $A_{20}^1$ | $A_{21}^1$ | $A_{22}^1$ | $A_{23}^1$ | $A_{24}^1$ | $A_{25}^1$ | $A_{26}^1$ | $A_{27}^1$ | $A_{28}^1$ | $A_{29}^1$ | $A_{30}^1$ | $A_{31}^1$ | $A_{32}^1$ | $A_{33}^1$ |
| $a_{11}$ | 0.83 | 0.93 | 0.73 | 0.63 | 0.95 | 0.88 | 0.93 | 0.95 | 0.75 | 0.98 | 0.93 | 0.99 | 0.68 | 0.93 | 0.88 | 0.96 | 0.68 | 0.78 | 0.95 | 0.83 |
| $a_{13}$ | 0.17 | 0.07 | 0.27 | 0.37 | 0.05 | 0.12 | 0.07 | 0.05 | 0.25 | 0.02 | 0.07 | 0.01 | 0.32 | 0.07 | 0.12 | 0.04 | 0.32 | 0.22 | 0.05 | 0.17 |
| $a_{22}$ | 0.99 | 0.95 | 0.79 | 0.89 | 0.79 | 0.89 | 0.83 | 0.95 | 0.85 | 0.89 | 0.79 | 0.93 | 0.94 | 0.98 | 0.75 | 0.95 | 0.85 | 0.99 |
| $a_{23}$ | 0.01 | 0.05 | 0.21 | 0.11 | 0.21 | 0.12 | 0.21 | 0.17 | 0.05 | 0.15 | 0.11 | 0.21 | 0.25 | 0.06 | 0.02 | 0.25 | 0.05 | 0.15 | 0.01 |
| $a_{31}$ | 0.40 | 0.50 | 0.40 | 0.60 | 0.50 | 0.30 | 0.50 | 0.49 | 0.45 | 0.37 | 0.40 | 0.55 | 0.48 | 0.75 | 0.65 | 0.50 | 0.25 | 0.60 | 0.70 | 0.40 |
| $a_{32}$ | 0.45 | 0.45 | 0.50 | 0.35 | 0.40 | 0.55 | 0.55 | 0.45 | 0.45 | 0.53 | 0.55 | 0.30 | 0.45 | 0.15 | 0.28 | 0.48 | 0.65 | 0.25 | 0.25 | 0.45 |
| $a_{33}$ | 0.15 | 0.05 | 0.10 | 0.05 | 0.10 | 0.15 | 0.05 | 0.06 | 0.10 | 0.05 | 0.15 | 0.07 | 0.10 | 0.07 | 0.02 | 0.10 | 0.15 | 0.05 | 0.15 |

Table C.4: First-Order SHFMM initial state transition probabilities (model 61-81)

|   | $A_{11}^3$ | $A_{12}^3$ | $A_{13}^3$ | $A_{14}^3$ | $A_{15}^3$ | $A_{16}^3$ | $A_{17}^3$ | $A_{18}^3$ | $A_{19}^3$ | $A_{20}^3$ | $A_{21}^3$ | $A_{22}^3$ | $A_{23}^3$ | $A_{24}^3$ | $A_{25}^3$ | $A_{26}^3$ | $A_{27}^3$ | $A_{28}^3$ | $A_{29}^3$ | $A_{30}^3$ | $A_{31}^3$ | $A_{32}^3$ | $A_{33}^3$ |
| $a_{11}$ | 0.77 | 0.82 | 0.93 | 0.96 | 0.91 | 0.99 | 0.75 | 0.76 | 0.97 | 0.81 | 0.75 | 0.88 | 0.95 | 0.79 | 0.89 | 0.91 | 0.99 | 0.74 |
| $a_{13}$ | 0.23 | 0.18 | 0.07 | 0.04 | 0.09 | 0.01 | 0.25 | 0.24 | 0.03 | 0.45 | 0.19 | 0.25 | 0.12 | 0.15 | 0.05 | 0.29 | 0.11 | 0.09 | 0.01 | 0.26 |
| $a_{22}$ | 0.89 | 0.91 | 0.92 | 0.87 | 0.91 | 0.89 | 0.91 | 0.94 | 0.77 | 0.89 | 0.81 | 0.97 | 0.78 | 0.87 | 0.74 | 0.71 | 0.79 | 0.87 | 0.79 | 0.74 |
| $a_{23}$ | 0.11 | 0.09 | 0.08 | 0.13 | 0.09 | 0.11 | 0.09 | 0.06 | 0.23 | 0.11 | 0.19 | 0.03 | 0.22 | 0.13 | 0.26 | 0.29 | 0.29 | 0.13 | 0.12 | 0.21 |
| $a_{31}$ | 0.50 | 0.37 | 0.50 | 0.75 | 0.36 | 0.40 | 0.71 | 0.61 | 0.52 | 0.62 | 0.73 | 0.64 | 0.72 | 0.52 | 0.47 | 0.79 | 0.69 | 0.62 | 0.52 | 0.23 |
| $a_{32}$ | 0.41 | 0.61 | 0.46 | 0.15 | 0.45 | 0.52 | 0.19 | 0.29 | 0.40 | 0.27 | 0.20 | 0.28 | 0.17 | 0.45 | 0.48 | 0.17 | 0.21 | 0.23 | 0.35 | 0.69 |
| $a_{33}$ | 0.09 | 0.02 | 0.04 | 0.10 | 0.19 | 0.08 | 0.10 | 0.10 | 0.08 | 0.11 | 0.07 | 0.08 | 0.11 | 0.03 | 0.05 | 0.04 | 0.10 | 0.15 | 0.13 | 0.08 |

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