A determinantal representation for derangement numbers

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Abstract

In the note, the author finds a representation for derangement numbers in terms of a tridiagonal determinant whose elements are the first few natural numbers.

\textbf{Keywords:} derangement number; determinantal representation; tridiagonal determinant.

In combinatorics, a derangement is a permutation of the elements of a set, such that no element appears in its original position. The number of derangements of a set of size \(n\) is called derangement number and sometimes denoted by \(!n\). The first ten derangement numbers \(!n\) for \(0 \leq n \leq 9\) are

1, 0, 1, 2, 9, 44, 265, 1854, 14833, 133496.

We now discover that derangement numbers \(!n\) can be beautifully expressed as a certain explicitly written down tridiagonal determinant. To the best of our knowledge, we have not seen such a representation in the context earlier.

\textbf{Theorem 1.} For \(n \in \{0\} \cup \mathbb{N}\), derangement numbers \(!n\) can be expressed by a tridiagonal \((n+1) \times (n+1)\) determinant

\[
!n = \begin{vmatrix}
-1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & -1 & 1 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & -2 & 2 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & n-3 & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & -(n-2) & n-2 & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & -(n-1) & n-1
\end{vmatrix}
\]

where

\[
e_{ij} = \begin{cases} 
1, & i-j = 1, \\
2, & i-j = 0, \\
1, & i-j = 0, \pm 1.
\end{cases}
\]

\textbf{Proof.} Once we write down the determinant, the proof of Theorem 1 can be made into a single line! Indeed, if the determinant written down in Theorem 1 is denoted by \(a_n\), then an induction immediately gives \(a_{n+1} = n(a_n + a_{n-1})\). This clearly produces derangement numbers \(!n\) which are determined by this recursion. Once discovered, the proof is just a single line. \hfill \qed

\textbf{Remark 1.} Recently, an alternative, although slightly complicated, proof of Theorem 1 was supplied in [1].

\textbf{References}