Properties and inequalities for the \((h_1, h_2)\)- and \((h_1, h_2, m)\)-GA-convex functions

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Abstract: In the paper, the authors introduce definitions of the \((h_1, h_2)\)-GA-convex functions and the \((h_1, h_2, m)\)-GA-convex functions, discuss some properties of these kinds of functions, establish some integral inequalities for these functions, and apply these inequalities to construct several more inequalities.

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1. Introduction
The following definitions are well known in the literature.

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The Inner Mongolia University for Nationalities locates at the Horqin (Khorchin, Horchin) grassland in China and was set up in 1958. The College of Mathematics is one of the eldest specialties and subjects in the university. Currently, there are 10 members in the research group of the Theory of Convexity and Applications. The members have introduced several new notions of convex functions and contributed much to the theory of convexity and applications.

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PUBLIC INTEREST STATEMENT
The theory of convex functions has important applications in many mathematical sciences. The notion of \(h\)-convex functions can be used to derive plenty of convex functions familiar to common mathematicians. In current paper, the authors extend the notion of \(h\)-convex functions, introduce a more general notion of \(h\)-convex functions, improve existed notions for convex functions, establish several inequalities for the general \(h\)-convex functions, and find their applications.
**Definition 1.1.** A function \( f: J \subseteq \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R} \) is said to be GA-convex if

\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)
\]

holds for all \( x, y \in I \) and \( t \in [0, 1] \).

**Definition 1.2.** (Toader, 1985) For \( f: [0, b] \rightarrow \mathbb{R}, b > 0 \) and \( m \in (0, 1] \), if

\[
f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)
\]

is valid for all \( x, y \in [0, b] \) and \( t \in [0, 1] \), then we say that \( f \) is an \( m \)-convex function on \([0, b]\).

**Definition 1.3.** (Hudzik & Maligranda, 1994) Let \( s \in (0, 1) \). A function \( f: [0, \infty) \rightarrow \mathbb{R}_0 \) is said to be \( s \)-convex in the second sense if

\[
f(sx + (1 - s)y) \leq sf(x) + (1 - s)f(y)
\]

holds for all \( x, y \in I \) and \( s \in [0, 1] \).

**Definition 1.4.** (Shuang, Yin, & Qi, 2013) Let \( f: I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_0 \) and \( s \in (0, 1) \). Then \( f \) is said to be an \( s \)-geometric-arithmetically convex function if

\[
f(tx^{1/s}y^{1-1/s}) \leq tf(x) + (1 - t)f(y)
\]

for \( x, y \in I \) and \( s \in [0, 1] \).

**Definition 1.5.** (Xi & Qi, 2015) For some \( s \in [-1, 1] \), a function \( f: I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R} \) is said to be extended \( s \)-convex if

\[
f(sx + (1 - s)y) \leq sf(x) + (1 - s)f(y)
\]

is valid for all \( x, y \in I \) and \( s \in (0, 1) \).

**Definition 1.6.** (Park, 2011) For \( (s, m) \in (0, 1)^2 \) and \( b > 0 \), a function \( f: [0, b] \rightarrow \mathbb{R} \) is said to be \( (s, m) \)-convex if

\[
f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)
\]

holds for all \( x, y \in I \) and \( s \in (0, 1) \).

**Definition 1.7.** (Varošanec, 2007) Let \( I, J \subseteq \mathbb{R} \) be intervals, \((0, 1) \subseteq J\), and \( h: J \rightarrow \mathbb{R} \) be a non-negative function such that \( h \neq 0 \). A function \( f: I \rightarrow \mathbb{R} \) is called \( h \)-convex, or say, \( f \in SX(h, I) \), if \( f \) is non-negative and

\[
f(tx + (1 - t)y) \leq h(tf(x) + h(1 - t)f(y)
\]

for all \( x, y \in I \) and \( t \in [0, 1] \). If the inequality (Equation 1.1) is reversed, then \( f \) is said to be \( h \)-concave, or say, \( f \in SV(h, I) \).

**Definition 1.8.** (Özdemir, Akdemir, & Set, 2011) Let \( J \subseteq \mathbb{R} \) be an interval, \((0, 1) \subseteq J\) and \( b > 0 \), \( h: J \rightarrow \mathbb{R} \) be a non-negative function such that \( h \neq 0 \). We say that \( f: [0, b] \rightarrow \mathbb{R} \) is an \((h, m)\)-convex function, or say, \( f \in SMX((h, m), [0, b]) \), if \( f \) is non-negative and

\[
f(tx + m(1 - t)y) \leq h(tf(x) + mh(1 - t)f(y)
\]

for all \( x, y \in [0, b] \) and \( m \in (0, 1] \) and \( m \in (0, 1] \). If the inequality (Equation 1.2) is reversed, then \( f \) is said to be \((h, m)\)-concave, or say, \( f \in SMV((h, m), [0, b]) \).
The following inequalities of Hermite–Hadamard type were established for some of the above convex functions.

**Theorem 1.1** (Dragomir & Toader, 1993, Theorem 2) Let \( f : \mathbb{R}_+ \to \mathbb{R} \) be an \( m \)-convex and \( m \in (0,1) \). If \( f \in L([a,b]) \) for \( 0 \leq a < b < \infty \), then

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + mf(b/m)}{2}, \frac{mf(a/m) + f(b)}{2} \right\}.
\]

**Theorem 1.2** (Dragomir, 2002, Theorem 2.2) Let \( f : \mathbb{R}_+ \to \mathbb{R} \) be an \( m \)-convex function with \( m \in (0,1) \). If \( f \in L([a,b]) \) for \( 0 \leq a < b < \infty \), then

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) + mf(x/m) \, dx \leq m + 1 \left\{ \frac{f(a) + f(b)}{2} + mf(a/m) + f(b/m) \right\}.
\]

**Theorem 1.3** (Sarikaya, Saglam, & Yildirim, 2008, Theorem 6) Let \( f \in S(h,I) \) and \( f \in L([a,b]) \) for \( a, b \in I \) with \( a < b \). Then

\[
\frac{1}{2h(1/2)} \int_a^b \frac{f(a+b)}{2} \, dx \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \int_0^1 h(t) \, dt.
\]

**Theorem 1.4** (Pinheiro, 2008, Theorem 4.2) If \( f \) is \( s \)-convex in the second sense and non-negative on \( I \) and \( x_1, x_2, \ldots, x_n \in I \) for \( n \geq 3 \) and some \( s \in (0,1) \), then

\[
\sum_{i=1}^{n} f(x_i) - f\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \frac{2^{s-1}(n^s - 1)}{n} \sum_{i,j} f\left(\frac{x_i + x_{i+1}}{2}\right).
\]

**Theorem 1.5** (Latif, 2010, Theorem 11) Let \( h \) be a non-negative super-multiplicative function. If \( f \in S(h,I) \) and \( x_1, \ldots, x_n \in I \), then

\[
\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) \geq \frac{1 - h(1/n)}{2h(1/2)} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right),
\]

where \( x_{n+1} = x_1 \). This inequality is reversed if \( f \in SV(h,I) \).


### 2. Definitions

We now introduce concepts of \((h_1, h_2)\)-GA-convex functions and \((h_1, h_2, m)\)-GA-convex functions.

**Definition 2.1.** Let \( h_i : [0,1] \to \mathbb{R}_+ \) such that \( h_i \neq 0 \) for \( i = 1, 2 \) and \( f : [a,b] \to \mathbb{R}_+ \). If

\[
f(x^y^{1-y}) \leq h_1(t)f(x) + h_2(1-t)f(y)
\]

for \( x, y \in I \) and \( t \in [0,1] \) then \( f \) is said to be an \((h_1, h_2)\)-geometric-arithmetic convex function or, simply speaking, an \((h_1, h_2)\)-GA-convex function. If Equation (2.1) is reversed, then \( f \) is said to be an \((h_1, h_2)\)-geometric-arithmetic concave function or, simply speaking, an \((h_1, h_2)\)-GA-concave function.

**Remark 1** If \( f \) is a decreasing and \((h_1, h_2)\)-GA-convex function on \( \mathbb{R}_+ \) and \( h_1(t) = h_2(t) = h(t) \) for \( t \in [0,1] \) then \( f \) is an \( h \) convex function on \( \mathbb{R}_+ \).
Remark 2  By Definition (2.1),

1. if \( h_1(t) = h_2(t) = h(t) \) for all \( t \in [0, 1] \), then \( f \) is an \( h \)-GA-convex function;
2. when \( h(t) = t^2 \) for \( t \in (0, 1) \) and \( s \in [-1, 1] \) an \( h \)-GA-convex function is reduced to an extended \( s \)-GA-convex function;
3. when \( h(t) = t \) for \( t \in [0, 1] \) an \( h \)-GA-convex function becomes a GA-convex function.

Definition 2.2. Let \( h,:[0,1] \rightarrow \mathbb{R}_+ \) and \( m: [0,1] \rightarrow (0,1) \) such that \( h \neq 0 \) for \( i = 1, 2 \). A function \( f: (0,b) \rightarrow \mathbb{R} \) is said to be \((h_1, h_2, m)\)-GA-convex, if

\[
f(x^y)^{1-t\ln t} \leq h_1(t)f(x) + m(t)h_2(1-t)f(y)
\]

for all \( x, y \in (0, b) \) and \( t \in [0, 1] \). If the inequality (Equation 2.2) is reversed, we say that \( f \) is \((h_1, h_2, m)\)-GA-concave.

Remark 3  By the above definitions, we have the following assertions.

1. If \( h_1(t) = h_2(t) = h(t) \) for all \( t \in [0, 1] \), then \( f : (0, b) \rightarrow \mathbb{R}_+ \) is an \((h, m)\)-GA-convex function;
2. Let \( f : (0, b) \rightarrow \mathbb{R}_+ \) be an \((h, m)\)-GA-convex function and \( m \in (0, 1) \) When \( h(t) = t \) for \( t \in [0, 1] \) the function \( f \) is said to be \( m \)-GA-convex;
3. If \( f : (0, b) \rightarrow \mathbb{R}_+ \) is an \((h, m)\)-GA-convex function, \( h(t) = t^2 \) for \( t \in (0, 1) \) and \( s \in [-1, 1] \) and \( m \in (0, 1) \), then \( f \) is extended \((s, m)\)-GA-convex;
4. If \( f : (0, b) \rightarrow \mathbb{R}_+ \) is an \((h, 1)\)-GA-convex function and \( m \in (0, 1) \), then it is \( h \)-GA-convex on \((0, b)\).

Example 2.1  Let \( f(x) = |\ln x| \) for \( x \in (0, 1) \) and \( m(t) = c(1 - t)^\varepsilon \) for \( t \in (0, 1) \) and \( 0 < c \leq 1 \), and some \( \varepsilon_0 \in \mathbb{R} \).

1. Let \( h_1(t) = t^{\varepsilon_1} \) and \( h_2(t) = t^{\varepsilon_2} \) for \( t \in (0, 1) \) and \( \varepsilon_1, \varepsilon_2 \in \mathbb{R} \). If \( \varepsilon_1, \varepsilon_2 \leq 1 \), then \( f \) is an decreasing and \((h_1, h_2, m)\)-GA-convex function on \((0, 1)\); if \( \varepsilon_1, \varepsilon_2 > 1 \), then \( f \) is an decreasing and \((h_1, h_2, m)\)-GA-concave function on \((0, 1)\).
2. In Definitions (1.8), letting \( m = 0.6 \), \( h(t) = t \) for all \( t \in [0, 1] \), \( x_0 = 0.5 \), \( y_0 = 0.9 \), and \( t_0 = 0.5 \) leads to

\[
f(t_0, x_0 + m(1 - t_0)y_0) - h(t_0)f(x_0) - mh(1 - t_0)f(y_0) > 0.
\]

This implies that \( f(x) = |\ln x| \) is not an \((h, m)\)-convex function on \((0, 1)\).

3. Properties
Now we discuss some properties of \((h_1, h_2, m)\)-GA-convex functions.

Theorem 3.1 Let \( h_1,:[0,1] \rightarrow \mathbb{R}_+ \) such that \( h_1 \neq 0 \) for \( i = 1, 2 \) and let \( f : [0, 1] \rightarrow \mathbb{R}_+ \).

1. If \( f : [0, 1] \rightarrow \mathbb{R}_+ \) is an \((h_1, h_2)\)-GA-convex function on \([0, 1]\), then \( h_1(t) + h_2(1-t) \geq 1 \) for \( t \in [0, 1]\);
2. If \( f : [0, 1] \rightarrow \mathbb{R}_+ \) is an \((h_1, h_2)\)-GA-concave function on \([0, 1]\), then \( h_1(t) + h_2(1-t) \leq 1 \) for \( t \in [0, 1]\).

Proof  If \( f : [0, 1] \rightarrow \mathbb{R}_+ \) is an \((h_1, h_2)\)-GA-convex function on \([0, 1]\), using the \((h_1, h_2)\)-GA-convexity of \( f \) on \([0, 1]\), we obtain

\[
f(x) = f(x^x^{1-x}) \leq h_1(t)f(x) + h_2(1-t)f(x) = (h_1(t) + h_2(1-t))f(x)
\]

for all \( x \in [0, 1] \) and \( t \in [0, 1] \).
The rest may be proved similarly. Theorem 3.1 is thus proved.

**THEOREM 3.2** Let $h_i: [0, 1] \to \mathbb{R}_0$ for $i = 1, 2, 3, 4, f_i: (0, b] \to \mathbb{R}_0$, and $m: [0, 1] \to (0, 1)$.

1. If $f$ is an $(h_1, h_2, m)$-GA-convex function on $(0, b]$, $h_1(t) \leq h_2(t)$, and $h_2(t) \leq h_4(t)$ for $t \in [0, 1]$, then $f$ is an $(h_3, h_4, m)$-GA-convex function on $(0, b]$.
2. If $f$ is an $(h_1, h_2, m)$-GA-concave function on $(0, b]$, $h_1(t) \geq h_2(t)$, and $h_2(t) \geq h_4(t)$ for $t \in [0, 1]$, then $f$ is an $(h_3, h_4, m)$-GA-concave function on $(0, b]$.

**Proof** Since $f$ is an $(h_1, h_2, m)$-GA-convex function on $(0, b]$ and $h_1(t) \leq h_2(t)$ for $t \in [0, 1]$, we have

$$f(x'y^{1-(t+m)}(t)) \leq h_1(t)f(x) + m(t)h_2(t)(1-t)f(x) \leq h_1(t)f(x) + m(t)h_4(t)(1-t)f(x)$$

for all $x \in (0, b]$ and $t \in [0, 1]$.

Since $f(x)$ is an $(h_1, m)$-GA-concave function on $(0, b]$ and $h_1(t) \geq h_2(t)$ for $t \in [0, 1]$, we have

$$f(x'y^{1-(t+m)}(t)) \geq h_1(t)f(x) + m(t)h_2(t)(1-t)f(x) \geq h_1(t)f(x) + m(t)h_4(t)(1-t)f(x)$$

for all $x \in (0, b]$ and $t \in [0, 1]$. The proof of Theorem 3.2 is complete.

**COROLLARY 3.2.1** Let $h_i: [0, 1] \to \mathbb{R}_0$ and $f_i: (0, b] \to \mathbb{R}_0$ for $1 \leq i \leq n$, and $m: [0, 1] \to (0, 1)$.

1. If $h(t) = \max_{1 \leq i \leq n} h_i(t)$ for $t \in [0, 1]$, then $\prod_{i=1}^n f_i$ is an $(h, m)$-GA-convex function on $(0, b]$.
2. If $h(t) = \min_{1 \leq i \leq n} h_i(t)$ for $t \in [0, 1]$, then $\prod_{i=1}^n f_i$ is an $(h, m)$-GA-concave function on $(0, b]$.

**Proof** This follows from Theorem 3.2 and induction on $n$.

**THEOREM 3.3** Let $h_i: [0, 1] \to \mathbb{R}_0$ such that $h_i \neq 0$ for $i = 1, 2, 3, 4$, $f_i: (0, b] \to \mathbb{R}_0$, $g: (0, d] \to g((0, d]) \subseteq (0, b]$, and $m \in (0, 1)$.

1. If $f$ is an increasing (or decreasing, respectively) and $(h_1, h_2, m)$-GA-convex function on $(0, b]$ and $u = g(x)$ is an $m$-geometrically convex (or concave, respectively) function on $(0, d]$, then $f \circ g$ is an $(h_1, h_2, m)$-GA-convex function on $(0, d]$.
2. If $f$ is an increasing (or decreasing, respectively) and $(h_1, h_2, m)$-GA-concave function on $(0, b]$ and $u = g(x)$ is an $m$-geometrically convex (or concave, respectively) function on $(0, d]$, then $f \circ g$ is an $(h_1, h_2, m)$-GA-concave function on $(0, d]$.

**Proof** When $f$ is a decreasing and $(h_1, h_2, m)$-GA-convex function on $(0, b]$,

1. if $u = g(x)$ is $m$-geometrically concave on $(0, d]$, then

   $$g(x'y^{m-1}) \geq (g(x))^{t}(g(y))^{m-1-t}$$

   for all $x, y \in (0, d]$ and $t \in [0, 1]$.
(2) if \( y = f(u) \) is decreasing and \((h_1, h_2, m)\)-GA-convex on \((0, b)\), then
\[
f(g(x^m)g(y)) \leq f(g(x))g(y) = h_1(t)f(g(x)) + mh_2(1-t)f(g(y))
\]
for all \( x, y \in (0, d) \) and \( t \in [0, 1] \).

Therefore, for all cases mentioned above, the composite \( f \circ g \) is an \((h_1, h_2, m)\)-GA-convex function on \([0, d]\).

The rest may be proved similarly. Theorem 3.3 is thus proved. \( \sqsubset \)

4. Jensen type inequalities

Now we are in a position to establish inequalities of Jensen type for \((h_1, h_2, m)\)-GA-convex functions.

**Theorem 4.1** Let \( h_i : [0, 1] \to \mathbb{R}_+ \) be functions such that \( h_i \neq 0 \) for \( i = 1, 2 \), let \( h_i(t_i)h_j(t_j) \leq h_i(t_i)h_j(t_j) \) for all \( t_i, t_j \in [0, 1] \) let \( h_2 \) be a super-multiplicative function and \( m : [0, 1] \to (0, 1] \) and let \( f : (0, b) \to \mathbb{R}_+ \) be an \((h_1, h_2, m)\)-GA-convex function on \((0, b)\). Then

\[
f\left( \prod_{i=1}^{n} x_i^{w_i} \prod_{j=1}^{m} m_{ij} \right) \leq h_1(w_1)f(x_1) + \sum_{i=2}^{n} \left( \prod_{j=1}^{m} m_{ij} \right) h_2(w_i)f(x_i)
\]

(4.1)

holds for all \( x_i \in (0, b) \), \( w_i > 0 \) such that \( \sum_{i=1}^{n} w_i = 1 \) and \( m(w_i) = 1 \).

If \( h_i(t_i)h_j(t_j) \geq h_i(t_i)h_j(t_j) \) for all \( t_i, t_j \in [0, 1] \), \( h \) is sub-multiplicative, and \( f \) is \((h_1, h_2, m)\)-GA-concave on \((0, b)\), then the inequality (Equation 4.1) is reversed.

**Proof** When \( n = 2 \), taking \( t = w_1 \) and \( 1-t = w_2 \) in Definition (2.2) means that the inequality (Equation 4.1) holds.

Suppose that the inequality (Equation 4.1) holds for \( n = k \), that is,

\[
f\left( \prod_{i=1}^{k} x_i^{w_i} \prod_{j=1}^{m} m_{ij} \right) \leq h_1(w_1)f(x_1) + \sum_{i=2}^{k} \left( \prod_{j=1}^{m} m_{ij} \right) h_2(w_i)f(x_i).
\]

(4.2)

When \( n = k + 1 \), letting \( \Delta_k = \sum_{i=2}^{k+1} w_i \), by Definition (2.2) and the hypothesis (Equation 4.2), we have

\[
f\left( \prod_{i=1}^{k+1} x_i^{w_i} \prod_{j=1}^{m} m_{ij} \right) = f\left( x_1^{w_1} \left( \prod_{i=2}^{k+1} x_i^{w_i/\Delta_k} \prod_{j=1}^{m} m_{ij}^{1/k} \right) \right)
\]

\[
\leq h_1(w_1)f(x_1) + m(w_1)h_2(\Delta_k)\left[ f\left( \left( \prod_{i=2}^{k+1} x_i^{w_i/\Delta_k} \prod_{j=1}^{m} m_{ij}^{1/k} \right) \right) \right]
\]

\[
\leq h_1(w_1)f(x_1) + m(w_1)h_2(\Delta_k)[h_2\left( \frac{w_2}{\Delta_k} \right) f(x_2)]
\]

\[
+ \sum_{i=2}^{k+1} \left( \prod_{j=1}^{m} m_{ij} \right) h_2\left( \frac{w_i}{\Delta_k} \right) f(x_i).
\]

Since \( h_2 \) is a super-multiplicative function, we obtain \( h_2(\Delta_k)h_2\left( \frac{w_i}{\Delta_k} \right) \leq h_2(w_i) \) for \( i = 1, 2, \ldots, n \). This implies that, when \( n = k + 1 \), the inequality (Equation 4.1) holds. By induction, Theorem 4.1 is proved. \( \sqsubset \)

**Remark 4** Under the conditions of Theorem 4.1, if \( f \) is a decreasing function on \((0, b)\) and \( m(t) = 1 \) for \( t \in [0, 1] \), then
holds for all $x_i \in (0, b)$ and $w_i > 0$ such that $\sum_{i=1}^{n} w_i = 1$.

**Corollary 4.1.1** Under the conditions of Theorem 4.1, if $w_1 = \cdots = w_n = \frac{1}{n}$, then

$$f \left( \prod_{i=1}^{n} x_i^{m_i} \right) \leq h_1 \left( \frac{1}{n} \right) f(x_1) + h_2 \left( \frac{1}{n} \right) \sum_{i=2}^{n} \left[ m \left( \frac{1}{n} \right) \right]^{i-1} f(x_i)$$

(4.3)

holds for all $x_i \in (0, b)$ for $i = 1, 2, \ldots, n$.

If $h_1(t_1)h_2(t_2) \geq h_2(t_1)h_2(t_2)$ for all $t_1, t_2 \in [0, 1]$, $h$ is sub-multiplicative, and $f$ is $(h_1, h_2, m)$-GA-concave on $(0, b)$, then the inequality (Equation 4.3) is reversed.

**Corollary 4.1.2** Let $h: [0, 1] \to \mathbb{R}_+$ be a super-multiplicative function such that $h \not\equiv 0$, $m \in (0, 1)$, and $f: (0, b) \to \mathbb{R}_+$ be an $(h, m)$-GA-convex function on $(0, b)$. Then the inequality

$$f \left( \prod_{i=1}^{n} x_i^{m_i} \right) \leq \sum_{i=1}^{n} m_i^{i-1} h(w_i) f(x_i)$$

(4.4)

holds for all $x_i \in (0, b)$ and $w_i > 0$ such that $\sum_{i=1}^{n} w_i = 1$. If $h$ is sub-multiplicative and $f$ is $(h, m)$-GA-concave on $(0, b)$, then the inequality (Equation 4.4) is reversed.

**Proof** This follows from Theorem 4.1 by putting $h_1(t) = h_2(t) = h(t)$ and $m(t) = m$ for all $t \in (0, 1)$ and $m \in (0, 1)$.

**Corollary 4.1.3** Let $h(t) = t^s$ for $t \in (0, 1)$ and $s \in [-1, 1]$, $f: (0, b) \to \mathbb{R}_+$ and $m \in (0, 1)$. Then $f$ is an $(s, m)$-GA-convex function on $(0, b)$ if and only if

$$f \left( \prod_{i=1}^{n} x_i^{m_i} \right) \leq \sum_{i=1}^{n} m_i^{i-1} w_i f(x_i)$$

for all $x_i \in (0, b)$ and $w_i > 0$ such that $\sum_{i=1}^{n} w_i = 1$.

5. Hermite–Hadamard type inequalities

Now we are in a position to establish some new Hermite–Hadamard type inequalities for $(h_1, h_2, m)$-GA-convex functions.

**Theorem 5.1** Let $h_i: [0, 1] \to \mathbb{R}_+$ for $i = 1, 2$, $m: [0, 1] \to (0, 1)$ and $f: \mathbb{R}_+ \to \mathbb{R}_+$ be an $(h_1, h_2, m)$-GA-convex function on $(0, b)$ such that $f \in L_1 \left( [a, b] / \left( m \frac{b}{\ln(t_2)} \right) \right)$ and $h_1, h_2 \in L_1((0, 1), f)$ for $0 < a < b$. Then

$$f (\sqrt{ab}) \leq \frac{h_1(1/2) \ln b - \ln a}{\ln b - \ln a} \int_0^b f(x) dx + \frac{m(1/2) h_2(1/2)}{\ln b - \ln a} \int_0^b f \left( \frac{x}{m(1/2)} \right) dx.$$

**Proof** Since $\sqrt{ab} = (a^t b^{1-t})^{1/2} (a^{1-t} b^t)^{1/2}$ for $0 \leq t \leq 1$, from the $(h_1, h_2, m)$-GA convexity of $f$ on $(0, b)$, we obtain

$$f (\sqrt{ab}) \leq h_1 \left( \frac{1}{2} \right) f (a^{1-t} b^t) + m \left( \frac{1}{2} \right) h_2 \left( \frac{1}{2} \right) f \left( \frac{a^{1-t} b^t}{m(1/2)} \right).$$

If replacing $a^{1-t} b^t$ and $a^t b^{1-t}$ for $0 \leq t \leq 1$ by $x$, then
\[
\int_0^1 f(a^{-1}b') \, dt = \frac{1}{\ln b - \ln a} \int_a^b f(x) \, dx \tag{5.1}
\]

and
\[
\int_0^1 f\left(\frac{a^{-1}b'}{m(1/2)}\right) \, dt = \frac{1}{\ln b - \ln a} \int_a^b f\left(\frac{x}{m(1/2)}\right) \, dx.
\tag{5.2}
\]

The proof of Theorem 5.1 is complete.

**THEOREM 5.2** Let \( h_i: [0, 1] \to \mathbb{R}, h \neq 0 \) for \( i = 1, 2, m \in \{0, 1\} \) and \( f: \mathbb{R}_+ \to \mathbb{R}_+ \) be an \((h_1, h_2, m)\)-GA-convex function on \((0, \frac{b}{m})\) such that \( f \in L_1([a, \frac{b}{m}] \) and \( h_1, h_2 \in L_1([0, 1]) \) for \( 0 < a < b \). Then
\[
\frac{1}{\ln b - \ln a} \int_a^b f(x) \, dx \leq \min \left\{ f(a) \int_0^1 h_1(t) \, dt + mf\left(\frac{b}{m}\right) \int_0^1 h_2(t) \, dt, \right.
\]
\[
\left. f(b) \int_0^1 h_1(t) \, dt + mf\left(\frac{a}{m}\right) \int_0^1 h_2(t) \, dt \right\}.
\]

If \( h_1(t) = h_2(t) = h(t) \) for all \( t \in [0, 1] \), we have
\[
\frac{1}{\ln b - \ln a} \int_a^b f(x) \, dx \leq \min \left\{ f(a) + mf\left(\frac{b}{m}\right), f(b) + mf\left(\frac{a}{m}\right) \right\} \int_0^1 h(t) \, dt.
\]

**Proof** Letting \( x = a^{-1}b' \) for \( 0 < t \leq 1 \), by the \((h_1, h_2, m)\)-GA-convexity of \( f \) and (Equation 5.1), we obtain
\[
\frac{1}{\ln b - \ln a} \int_a^b f(x) \, dx = \int_0^1 f(a^{-1}b') \, dt \leq \min \left\{ f(a) \int_0^1 h_1(t) \, dt \right.
\]
\[
\left. + mf\left(\frac{b}{m}\right) \int_0^1 h_2(t) \, dt, f(b) \int_0^1 h_1(t) \, dt + mf\left(\frac{a}{m}\right) \int_0^1 h_2(t) \, dt \right\}.
\]

The proof of Theorem 5.2 is complete.

**COROLLARY 5.2.1** Let \( h_1(t) = t^{s_1} \) and \( h_2(t) = t^{s_2} \) for all \( t \in (0, 1) \), let \( s_1, s_2 \in (-1, 1) \) and \( m \in \{0, 1\} \), and let \( f: \mathbb{R}_+ \to \mathbb{R}_+ \) be an \((h_1, h_2, m)\)-GA-convex function on \((0, \frac{b}{m})\) such that \( f \in L_1([a, \frac{b}{m}] \) for \( 0 < a < b \). Then
\[
\frac{1}{\ln b - \ln a} \int_a^b f(x) \, dx \leq \min \left\{ f(a) \int_0^1 h_1(t) \, dt + mf\left(\frac{b}{m}\right), f(b) \int_0^1 h_1(t) \, dt + mf\left(\frac{a}{m}\right) \right\}.
\]

**THEOREM 5.3** Let \( h_i: [0, 1] \to \mathbb{R}, h \neq 0 \) for \( i = 1, 2, m \in \{0, 1\} \) and \( f: \mathbb{R}_+ \to \mathbb{R}_+ \) be an \((h_1, h_2, m)\)-GA-convex function on \((0, \frac{b}{m})\) such that \( f \in L_1([a, \frac{b}{m}] \) and \( h_1, h_2 \in L_1([0, 1]) \) for \( 0 < a < b \). Then
\[
f\left(\sqrt{ab}\right) \leq \frac{h_1(1/2)}{\ln b - \ln a} \int_0^b f(x) \, dx + \frac{mh_2(1/2)}{\ln b - \ln a} \int_0^b f\left(\frac{x}{m}\right) \, dx
\]
\[
\leq \min \left\{ h_1\left(\frac{1}{2}\right) f(a) + mh_2\left(\frac{1}{2}\right) f\left(\frac{a}{m}\right) \int_0^1 h_1(t) \, dt,
\right.
\]
\[
h_1\left(\frac{1}{2}\right) f(b) + mh_2\left(\frac{1}{2}\right) f\left(\frac{b}{m}\right) \int_0^1 h_1(t) \, dt,
\]
\[
\left. h_1\left(\frac{1}{2}\right) f(a) + mh_2\left(\frac{1}{2}\right) f\left(\frac{a}{m}\right) \int_0^1 h_2(t) \, dt \right\}.
\]

**Proof** From the \((h_1, h_2, m)\)-GA convexity of \( f \) on \((0, \frac{b}{m})\), we obtain
\[
f(\sqrt{ab}) \leq h_1\left(\frac{1}{2}\right)f(a\cdot b^{1-t}) + mh_2\left(\frac{1}{2}\right)f\left(\frac{a^{1-t}b^t}{m}\right)
\]
\[
\leq \min\left\{h_1\left(\frac{1}{2}\right)\left[h_1(t)f(a) + mh_2(1-t)f\left(\frac{b}{m}\right)\right] + mh_2\left(\frac{1}{2}\right)\left[h_1(1-t)f\left(\frac{a}{m}\right) + mh_2(t)f\left(\frac{b}{m^2}\right)\right],
\right.
\]
\[
\left.h_1\left(\frac{1}{2}\right)\left[h_1(1-t)f(b) + mh_2(t)f\left(\frac{a}{m}\right)\right] + mh_2\left(\frac{1}{2}\right)\left[h_1(t)f\left(\frac{b}{m}\right) + mh_2(1-t)f\left(\frac{a}{m}\right)\right]\right\}.
\]

Substituting \(a^{1-t}b^t\) and \(a\cdot b^{1-t}\) for \(0 \leq t \leq 1\) by \(x\) and integrating on both sides of the above inequality with respect to \(t \in [0,1]\) lead to
\[
f(\sqrt{ab}) \leq \frac{h_1(1/2)}{\ln b - \ln a} \int_0^b f(x)dx + \frac{mh_2(1/2)}{\ln b - \ln a} \int_0^b f\left(\frac{x}{m}\right)dx
\]
\[
\leq \min\left\{\left[h_1\left(\frac{1}{2}\right)f(a) + mh_2\left(\frac{1}{2}\right)f\left(\frac{b}{m}\right)\right]\int_0^1 h_1(t)dt + m\left[h_1\left(\frac{1}{2}\right)f\left(\frac{b}{m}\right) + mh_2\left(\frac{1}{2}\right)f\left(\frac{a}{m^2}\right)\right]\int_0^1 h_2(t)dt,
\right.
\]
\[
\left[h_1\left(\frac{1}{2}\right)f(b) + mh_2\left(\frac{1}{2}\right)f\left(\frac{a}{m}\right)\right]\int_0^1 h_2(t)dt + m\left[h_1\left(\frac{1}{2}\right)f\left(\frac{a}{m}\right) + mh_2\left(\frac{1}{2}\right)f\left(\frac{b}{m^2}\right)\right]\int_0^1 h_1(t)dt\right\}.
\]

Theorem 5.3 is proved.

**Corollary 5.3.1** Let \(h:[0,1] \to \mathbb{R}_+\) \(h\neq 0\) and \(m \in (0,1)\) and \(f: \mathbb{R}_+ \to \mathbb{R}_+\) be an \((h, m)-GA\)-convex function on \((0, b/m^2]\) such that \(f \in L_1\left([0, b/m]\right)\) and \(h \in L_1([0,1])\) for \(0 < a < b\). Then
\[
f(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a} \int_0^b f(x)dx + mf\left(\frac{x}{m}\right)\int_0^1 h(t)dt
\]
\[
\leq \min\left\{f(a) + mf\left(\frac{a}{m}\right) + mf\left(\frac{b}{m}\right) + m^2f\left(\frac{b}{m^2}\right),
\right.
\]
\[
2mf\left(\frac{a}{m}\right) + f(b) + m^2f\left(\frac{b}{m^2}\right), f(a) + m^2f\left(\frac{a}{m^2}\right) + 2mf\left(\frac{b}{m}\right),
\right.
\]
\[
mp\left(\frac{a}{m}\right) + m^2f\left(\frac{a}{m^2}\right) + f(b) + m^2f\left(\frac{b}{m^2}\right)\right\}\int_0^1 h(t)dt.
\]

**Proof** This can be derived from letting \(h_1(t) = h_2(t) = h(t)\) for all \(t \in [0,1]\) and considering the symmetry between \(a\) and \(b\) in Theorem 4.1.

**Corollary 5.3.2** Under the conditions of Corollary 5.3.1, if \(h(t) = t^s\) for \(t \in (0,1)\) and \(s \in (-1, 1]\), then
\[
2^sf(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a} \int_0^b f(x)dx + mf\left(\frac{x}{m}\right)\int_0^1 h(t)dt
\]
\[
\leq \frac{1}{s+1} \min\left\{f(a) + mf\left(\frac{a}{m}\right) + mf\left(\frac{b}{m}\right) + m^2f\left(\frac{b}{m^2}\right),
\right.
\]
\[
2mf\left(\frac{a}{m}\right) + f(b) + m^2f\left(\frac{b}{m^2}\right), f(a) + mf\left(\frac{a}{m^2}\right)
\]
\[
+ 2mf\left(\frac{b}{m}\right), mf\left(\frac{a}{m}\right) + m^2f\left(\frac{a}{m^2}\right) + f(b) + mf\left(\frac{b}{m}\right)\right\}\int_0^1 h(t)dt.
\]
THEOREM 5.4 Let $h_i: [0, 1] \to \mathbb{R}_+$ such that $h_i \not\equiv 0$ for $i = 1, 2$, $m: [0, 1] \to (0, 1]$ and $f, g: \mathbb{R}_+ \to \mathbb{R}_+$ are $(h_1, h_2, m)$-GA-convex functions on $(0, \frac{b}{m(1/2)})$ such that $fg \in L_1([a, \frac{b}{m(1/2)})$ for $0 < a < b$. Then

$$f(\sqrt{ab})g(\sqrt{ab}) \leq \frac{h_1(1/2)}{ln b - ln a} \int_0^b (f(x)g(x)) dx$$

$$+ \frac{m(1/2)h_1(1/2)h_2(1/2)}{ln b - ln a} \int_0^b \left( f\left(\frac{h_1(1/2)}{m(1/2)}x\right)g(x) + f(x)g\left(\frac{x}{m(1/2)}\right) \right) dx$$

$$+ \frac{m(1/2)h_2(1/2)}{ln b - ln a} \int_0^b f\left(\frac{x}{m(1/2)}\right)g\left(\frac{x}{m(1/2)}\right) dx.$$  

(5.3)

Proof Using the $(h_1, h_2, m)$-GA convexity of $f$ and $g$ on $(0, \frac{b}{m(1/2)})$, we obtain

$$f(\sqrt{ab})g(\sqrt{ab}) \leq \left[ h_1(1/2)\frac{i(a'b^{1-t})}{2} + m(1/2)h_2(1/2)\frac{i(a'b^{1-t})}{m(1/2)} \right]$$

$$\times \left[ h_1(1/2)\frac{i(a'b^{1-t})}{2} + m(1/2)h_2(1/2)\frac{i(a'b^{1-t})}{m(1/2)} \right].$$  

(5.4)

Letting $x = a'b^{1-t}$ and $x = a'b^{1-t}$ for $t \in [0, 1]$ and integrating the inequality (Equation 5.4) on $[0, 1]$ with respect to $t$, we arrive at the inequality (Equation 5.3). Theorem 5.4 is thus proved.

THEOREM 5.5 Let $h_i: [0, 1] \to \mathbb{R}_+$ $h_i \not\equiv 0$ for $i = 1, 2$, $m_1, m_2 \in (0, 1]$ $f, g: \mathbb{R}_+ \to \mathbb{R}_+$ If $f$ is an $(h_1, h_2, m_1)$-GA-convex function on $(0, \frac{b}{m_1})$ $g$ is an $(h_1, h_2, m_2)$-GA-convex function on $(0, \frac{b}{m_2})$ and $fg \in L_1([a, b])$ and $h_1, h_2 \in L_1([0, 1])$ for $0 < a < b$, then

$$\frac{1}{ln b - ln a} \int_0^b f(x)g(x) dx$$

$$\leq f(a)g(a) \int_0^1 h_1^2(t) dt + m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right) \int_0^1 h_2^2(t) dt$$

$$+ \left[ m_2f(a)g\left(\frac{b}{m_2}\right) + m_1f\left(\frac{b}{m_1}\right)g(a) \right] \int_0^1 h_1(t)h_2(1-t) dt.$$  

(5.5)

Proof Let $x = a'b^{1-t}$ for $0 \leq t \leq 1$. By the $(h_1, h_2, m)$-GA-convexity of $f$ and $g$, we have

$$\frac{1}{ln b - ln a} \int_0^b f(x)g(x) dx = \int_0^1 f(a'b^{1-t})g(a'b^{1-t}) dt$$

$$\leq \int_0^1 h_1(t) f(a) + m_1h_2(1-t)f\left(\frac{b}{m_1}\right) \left[ h_1(t)g(a) + m_2h_2(1-t)g\left(\frac{b}{m_2}\right) \right] dt$$

$$= f(a)g(a) \int_0^1 h_1^2(t) dt + m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right) \int_0^1 h_2^2(t) dt$$

$$+ \left[ m_2f(a)g\left(\frac{b}{m_2}\right) + m_1f\left(\frac{b}{m_1}\right)g(a) \right] \int_0^1 h_1(t)h_2(1-t) dt.$$  

The proof of Theorem 5.5 is complete.

COROLLARY 5.5.1 Under the conditions of Theorem 5.5, if $h_1(t) = h_2(t) = h(t)$ for all $t \in [0, 1]$, then

$$\frac{1}{ln b - ln a} \int_0^b f(x)g(x) dx \leq \left[ f(a)g(a) + m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right) \right]$$

$$\times \int_0^1 h_1^2(t) dt + \left[ m_2f\left(\frac{b}{m_2}\right)g(a) + m_1f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right) \right] \int_0^1 h(t)h(1-t) dt.$$  

In particular, if $h(t) = t^s$ for $t \in (0, 1)$, $s \in (-\frac{1}{2}, 1)$ and $m_1 = m_2 = m$, then
\[ \frac{1}{\ln b - \ln a} \int_0^b f(x)g(x) \, dx \leq \frac{1}{2s + 1} \left[ f(\alpha)g(\alpha) + m^2f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right) \right] \\
+ mB(s + 1, s + 1) \left[ f(\alpha)g\left(\frac{b}{m}\right) + f\left(\frac{b}{m}\right)g(\alpha) \right], \]

where \( B \) denotes the well-known Beta function.

6. Applications

In what follows we will apply theorems and corollaries in the above section to establish inequalities for \((h_1, h_2, m)\)-GA-convex functions.

**Theorem 6.1** Under the conditions of Theorem 4.1, if \( n, k \in \mathbb{N} \) with \( n \geq 3 \) and \( 2 \leq k \leq n \). Then, for all \( x_i \in (0, b) \) and \( w > 0 \) such that \( \sum_{i=1}^{n} w_i = 1 \), \( m(w_0) = 1 \), and \( x_{n+1} = x_1, \ldots, x_{2n-1} = x_{n+1} \)

\( f \) is \((h_1, h_2, m)\)-GA-concave on \((0, b)\), then the inequalities (Equations 6.1 and 6.2) are reversed.

| (1) when \( 1 - h_2\left(\frac{1}{n}\right) \neq 0 \), we have |
| \( \sum_{i=1}^{n} f\left(\prod_{p=1}^{k-1} x_p^{w_{p-1} \cdots m(w_0)}\right) \)
| \( \leq \left[ h_1(w_1) + \left( \frac{1}{n} \right) \sum_{i=1}^{n} f(x_i) + \left( \frac{1}{n} \right) \sum_{i=1}^{n} f\left(\prod_{p=1}^{k-1} x_p^{w_{p-1}}\right) \right]^n \)
| \( \times h_2(w_1) \left\{ h_1\left(\frac{1}{n}\right) - 1 \right\} \sum_{i=1}^{n} f(x_i) - \left[ \frac{1}{n} \right] \sum_{i=1}^{n} f\left(\prod_{p=1}^{k-1} x_p^{w_{p-1}}\right) \}\right) \right) \]

| (2) when \( 1 - h_2\left(\frac{1}{n}\right) = 0 \), we have |
| \( \sum_{i=1}^{n} f\left(\prod_{p=1}^{k-1} x_p^{w_{p-1} \cdots m(w_0)}\right) \)
| \( \leq \left[ h_1(w_1) + \left( \frac{1}{n} \right) \sum_{i=1}^{n} f(x_i) + \left( \frac{1}{n} \right) \sum_{i=1}^{n} f\left(\prod_{p=1}^{k-1} x_p^{w_{p-1}}\right) \right]^n \)
| \( \times h_2(w_1) \left\{ h_1\left(\frac{1}{n}\right) - 1 \right\} \sum_{i=1}^{n} f(x_i) - \left[ \frac{1}{n} \right] \sum_{i=1}^{n} f\left(\prod_{p=1}^{k-1} x_p^{w_{p-1}}\right) \}\right) \right) \]

If \( h_1(t_1, h_2) \geq h_1(t_2, t_3) \) for all \( t_1, t_2 \in (0, b) \) \( h_1 \) is sub-multiplicative, and \( f \) is \((h_1, h_2, m)\)-GA-concave on \((0, b)\), then the inequalities (Equations 6.1 and 6.2) are reversed.

**Proof** Using the inequality (Equation 4.1), it follows that

\[ \sum_{i=1}^{n} f\left(\prod_{p=1}^{k-1} x_p^{w_{p-1} \cdots m(w_0)}\right) \leq \left[ h_1(w_1) - h_2(w_1) \right] \sum_{i=1}^{n} f(x_i) + \left( \frac{1}{n} \right) \sum_{i=1}^{n} f\left(\prod_{p=1}^{k-1} x_p^{w_{p-1}}\right) \sum_{i=1}^{n} f(x_i). \]

When \( h_2\left(\frac{1}{n}\right) \neq 1 \), we have

\[ \sum_{i=1}^{n} f\left(\prod_{p=1}^{k-1} x_p^{w_{p-1} \cdots m(w_0)}\right) \leq \left[ h_1(w_1) - h_2(w_1) \right] \sum_{i=1}^{n} f(x_i) + \left[ 1 - h_2\left(\frac{1}{n}\right) \right] \sum_{i=1}^{n} f(x_i) - \left[ \frac{1}{n} \right] \sum_{i=1}^{n} f(x_i). \]
By the inequality (Equation 4.3), it is easy to see that

\[
1 - h_2 \left( \frac{1}{n} \right) \sum_{i=1}^{n} f \left( \prod_{p=1}^{k+i-1} x_p^{w_{p-1}} \prod_{j=0}^{\lfloor \frac{i}{k} \rfloor} x_p^{w_j} \right) \\
\leq 1 - h_2 \left( \frac{1}{n} \right) \left[ h_1(w_1) - h_2(w_1) \right] \sum_{i=1}^{n} f(x_i) \\
+ \left( \sum_{p=0}^{k-1} \sum_{h=0}^{n-1} \prod_{j=0}^{h} m(w_j) h_2(w_{p+h+1}) \right) \\
\left\{ \sum_{i=1}^{n} f(x_i) + \frac{h_1 \left( \frac{1}{p} \right) - h_2 \left( \frac{1}{p} \right)}{\sum_{p=0}^{h} \prod_{j=0}^{h} m \left( \frac{1}{p+1} \right) \prod_{j=0}^{h} m \left( \frac{1}{p+1} \right)} \sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f \left( \prod_{p=1}^{k+i-1} x_p^{w_{p-1}} \prod_{j=0}^{\lfloor \frac{i}{k} \rfloor} x_p^{w_j} \right) \right\}.
\]

When \( h_2 \left( \frac{1}{p} \right) = 1 \), we obtain

\[
\sum_{i=1}^{n} f \left( \prod_{p=1}^{k+i-1} x_p^{w_{p-1}} \prod_{j=0}^{\lfloor \frac{i}{k} \rfloor} x_p^{w_j} \right) \\
\leq \left[ h_1(w_1) + \sum_{p=0}^{k-1} \sum_{j=0}^{n-1} \prod_{p=0}^{j} m(w_j) h_2(w_{p+j+1}) \right] \sum_{i=1}^{n} f(x_i) - \left( \sum_{p=0}^{n-1} \prod_{j=0}^{h} m \left( \frac{1}{p+1} \right) \right)^{1-p} \\
\times h_2(w_1) \left\{ h_1 \left( \frac{1}{n} \right) - 1 \right\} \sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f \left( \prod_{p=1}^{k+i-1} x_p^{w_{p-1}} \prod_{j=0}^{\lfloor \frac{i}{k} \rfloor} x_p^{w_j} \right) \right\}.
\]

The proof of Theorem 6.1 is complete.

**COROLLARY 6.1.1** Under the conditions of Theorem 6.1, if \( w_1 = \cdots = w_k = \frac{1}{k} \),

1. when \( 1 - h_2 \left( \frac{1}{n} \right) \neq 0 \), we have

\[
1 - h_2 \left( \frac{1}{n} \right) \sum_{i=1}^{n} f \left( \prod_{p=1}^{k+i-1} x_p^{w_p} \right) \\
\leq \left[ \prod_{p=1}^{k+i-1} x_p^{w_p} \right] \sum_{i=1}^{n} f(x_i) + \frac{h_1 \left( \frac{1}{p} \right) - h_2 \left( \frac{1}{p} \right)}{\sum_{p=0}^{h} \prod_{j=0}^{h} m \left( \frac{1}{p+1} \right) \prod_{j=0}^{h} m \left( \frac{1}{p+1} \right)} \sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f \left( \prod_{p=1}^{k+i-1} x_p^{w_p} \right) \right\}.
\]

2. when \( 1 - h_2 \left( \frac{1}{n} \right) = 0 \), we have

\[
\sum_{i=1}^{n} f \left( \prod_{p=1}^{k+i-1} x_p^{w_p} \right) \\
\leq \left[ h_1 \left( \frac{1}{k} \right) + \frac{h_2 \left( \frac{1}{k} \right)}{\sum_{p=0}^{h} \prod_{j=0}^{h} m \left( \frac{1}{p+1} \right) \prod_{j=0}^{h} m \left( \frac{1}{p+1} \right)} \sum_{i=1}^{n} f(x_i) + \left( \sum_{p=0}^{n-1} \prod_{j=0}^{h} m \left( \frac{1}{p+1} \right) \right)^{1-p} \\
\times h_2 \left( \frac{1}{k} \right) \left\{ h_1 \left( \frac{1}{n} \right) - 1 \right\} \sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f \left( \prod_{p=1}^{k+i-1} x_p^{w_p} \right) \right\}.
\]
If \( h_1(t_1)h_2(t_2) \geq h_1(t_1)h_2(t_2) \) for all \( t_1, t_2 \in [0, 1] \), \( h \) is sub-multiplicative, and \( f \) is \((h_1, h_2, m)\)-GA-concave on \((0, b)\), then the inequalities (Equations 6.3 and 6.4) are reversed.

**Corollary 6.1.2** Under the conditions of Theorem 6.1, if \( w_1 = w_2 = \cdots = w_k = \frac{1}{k} \) and \( m(t) = 1 \) for any \( t \in [0, 1] \),

1. when \( 1 - h_2 \left( \frac{1}{k} \right) \neq 0 \), we have

\[
\frac{1 - h_2 \left( \frac{1}{k} \right)}{kh_2 \left( \frac{1}{k} \right)} \left[ \sum_{i=1}^{n} f \left( \prod_{p=i}^{k-i} \frac{x_p^{ \frac{1}{n} }}{n} \right) \right] \leq \left[ 1 - h_2 \left( \frac{1}{k} \right) \right] \left[ h_1 \left( \frac{1}{k} \right) - h_2 \left( \frac{1}{k} \right) \right]
\]

\[
\times \sum_{i=1}^{n} f(x_i) + \frac{1}{n} \left[ n + h_1 \left( \frac{1}{n} \right) \right] \sum_{i=1}^{n} f(x_i) - f \left( \prod_{i=1}^{n} x_i^{ \frac{1}{n} } \right); \tag{6.5}
\]

2. when \( 1 - h_2 \left( \frac{1}{k} \right) = 0 \), we have

\[
\sum_{i=1}^{n} f \left( \prod_{p=i}^{k-i} \frac{x_p^{ \frac{1}{n} }}{n} \right) \leq h_1 \left( \frac{1}{k} \right) + kh_2 \left( \frac{1}{k} \right) \sum_{i=1}^{n} f(x_i)
\]

\[
+ h_2 \left( \frac{1}{k} \right) \left[ \frac{1}{n} h_1 \left( \frac{1}{n} \right) - 1 \right] \sum_{i=1}^{n} f(x_i) - f \left( \prod_{i=1}^{n} x_i^{ \frac{1}{n} } \right). \tag{6.6}
\]

If \( h_1(t_1)h_2(t_2) \geq h_1(t_1)t_2 \) for all \( t_1, t_2 \in [0, 1] \), \( h \) is sub-multiplicative, and \( f \) is \((h_1, h_2, m)\)-GA-concave on \((0, b)\), then the inequalities (Equation 6.5 and 6.6) are reversed.

**Corollary 6.1.3** Under the conditions of Theorem 6.1, if \( h_1(t) = t^i \) and \( h_2(t) = t^i \) for all \( t \in (0, 1) \) and \( s_1, s_2 \in [-1, 1] \), then

1. when \( s_1 \in [-1, 1], s_2 \neq 0, \) and \( s_1 \geq s_2 \), we have

\[
\frac{1 - n^{-s_1}}{k^{-s_1}} \sum_{i=1}^{n} f \left( \prod_{p=i}^{k-i} \frac{x_p^{ \frac{1}{n} (1-m)^{i-1} } }{n} \right) \leq \left[ 1 - n^{-s_1} \right] \left[ k^{-s_1} - k^{-2s_1} \right]
\]

\[
\times \sum_{i=1}^{n} f(x_i) + \sum_{i=0}^{k-i} \left[ m \left( \frac{1}{k} \right) \right]^{i} \left[ \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n-1} h_1 - n^{-s_1} \right] \sum_{i=1}^{n} f(x_i)
\]

\[
\times \sum_{i=1}^{n} f(x_i) - \frac{n^{n-i} f \left( \prod_{i=1}^{n-i} x_p^{ \frac{1}{n} (1-m)^{i-1} } \right) }{\sum_{i=1}^{n} f(x_i)} \tag{6.7}
\]

2. when \( 1 \geq s_1 \geq s_2 = 0 \), we have

\[
\sum_{i=1}^{n} f \left( \prod_{p=i}^{k-i} \frac{x_p^{ \frac{1}{n} (1-m)^{i-1} } }{n} \right) \leq \left[ w_1^{s_1} + \sum_{i=0}^{k-i} \left( \prod_{p=0}^{i-1} m(w_j) \right) \right] \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n-1} m \left( \frac{1}{n} \right) \left[ n^{-s_1} \right]^{-1}
\]

\[
\times \left[ \left( n^{-s_1} - 1 \right) \sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f \left( \prod_{p=i}^{n-i} x_p^{ \frac{1}{n} (1-m)^{i-1} } \right) \right] \tag{6.8}
\]

If \( f \) is \((h_1, h_2, m)\)-GA-concave on \((0, b)\) and \( s_1 \leq s_2 \), then the inequalities (Equations 6.7 and 6.8) are reversed.

**Corollary 6.1.4** Under the conditions of Theorem 6.1, if \( w_1 = \cdots = w_k = \frac{1}{k} \), \( h_1(t) = h_2(t) = h(t) \), and \( m(t) = 1 \) for \( t \in [0, 1] \) and \( h \left( \frac{1}{n} \right) \leq 1 \) and \( f \) is a decreasing function on \((0, b)\), then
Theorem 6.2

Under the conditions of Theorem 6.1, if \( \epsilon_1, \ldots, \epsilon_k \in \mathbb{N} \) then

(1) when \( h_{1\frac{1}{n}} \neq 0 \), we have

\[
\frac{1 - h_{1\frac{1}{n}}}{k h_{1\frac{1}{n}}} \sum_{i=1}^{n} \sum_{p=1}^{k} x_{p}^{\frac{i}{m}} \leq \frac{1 - h_{1\frac{1}{n}}}{k h_{1\frac{1}{n}}} \sum_{i=1}^{n} \left( \prod_{p=1}^{k} x_{p}^{\frac{i}{m}} \right) \\
\leq \sum_{i=1}^{n} f(x_{i}) - f \left( \prod_{i=1}^{n} x_{i}^{\frac{1}{m}} \right) \leq \sum_{i=1}^{n} f(x_{i}) - f \left( \frac{1}{n} \sum_{i=1}^{n} x_{i} \right); \tag{6.9}
\]

(2) when \( h_{1\frac{1}{n}} = 1 \), we have

\[
\frac{1}{h_{1\frac{1}{n}}} \sum_{i=1}^{n} \sum_{p=1}^{k} x_{p}^{\frac{i}{m}} \leq \frac{1}{h_{1\frac{1}{n}}} \sum_{i=1}^{n} \left( \prod_{p=1}^{k} x_{p}^{\frac{i}{m}} \right) \\
\leq (1 + k) \sum_{i=1}^{n} f(x_{i}) - f \left( \prod_{i=1}^{n} x_{i}^{\frac{1}{m}} \right) \leq (1 + k) \sum_{i=1}^{n} f(x_{i}) - f \left( \frac{1}{n} \sum_{i=1}^{n} x_{i} \right).
\]

Theorem 6.2

Under the conditions of Theorem 6.1, if \( \epsilon_1, \ldots, \epsilon_k \in \mathbb{N} \) then

(1) when \( 1 - h_{2\frac{1}{n}} \neq 0 \), we have

\[
\frac{1 - h_{2\frac{1}{n}}}{n - 1} \sum_{1 \leq i < j < \infty} \sum_{p=1}^{k} f \left( \prod_{p=1}^{k} x_{p}^{\frac{i}{m} + \frac{j}{m}} \right) \\
\leq \left[ 1 - h_{2\frac{1}{n}} \left( \frac{1}{n} \right) \right] \left| h_{1}(w_{1}) - h_{2}(w_{2}) \right| \sum_{i=1}^{n} f(x_{i}) \\
+ \left[ \frac{\sum_{p=0}^{k} \sum_{j=0}^{n} m(w_{j}) h_{2}(w_{p+j}) f \left( \prod_{p=1}^{k} x_{p}^{\frac{i}{m} + \frac{j}{m}} \right) }{\sum_{p=0}^{n-1} \left( m \left( \frac{1}{n} \right) \right)^{p}} \sum_{i=1}^{n} f(x_{i}) \right]; \tag{6.10}
\]

(2) when \( 1 - h_{2\frac{1}{n}} = 0 \), we have

\[
\frac{1}{n - 1} \sum_{1 \leq i < j < \infty} \sum_{p=1}^{k} f \left( \prod_{p=1}^{k} x_{p}^{\frac{i}{m} + \frac{j}{m}} \right) \\
\leq \left[ h_{1}(w_{1}) + \sum_{p=0}^{k} \sum_{j=0}^{n} m(w_{j}) h_{2}(w_{p+j}) \right] \sum_{i=1}^{n} f(x_{i}) + \frac{h_{1}(w_{1})}{\sum_{p=0}^{n-1} \left( m \left( \frac{1}{n} \right) \right)^{p}} \\
\times \left\{ h_{1} \left( \frac{1}{n} \right) - 1 \right\} \sum_{i=1}^{n} f(x_{i}) - f \left( \prod_{p=1}^{k} x_{p}^{\frac{i}{m} + \frac{j}{m}} \right) \}; \tag{6.11}
\]

where \( \epsilon_{k+1} = \epsilon_{1}, \ldots, \epsilon_{2k-1} = \epsilon_{k-1} \).

If \( h_{1}(t_{1}), h_{2}(t_{2}) \geq h_{2}(t_{1} t_{2}) \) for all \( t_{1}, t_{2} \in [0, 1] \), \( h_{2} \) is sub-multiplicative, and \( f \) is \( (h_{1}, h_{2}, m) \)-GA-concave on \( (0, b) \), then the inequalities (Equations 6.10 and 6.11) are reversed.

Proof Using the inequality (Equation 4.1), it follows that
The proof of Theorem 6.2 is complete.

\[
\frac{1}{n-1} \sum_{k=2}^{n} \sum_{i=1}^{k-1} f \left( \prod_{p=1}^{i-1} x_p^{w_{p+1} m(i)} \right) \leq \left| h_1(w_1) - h_2(w_1) \right| \sum_{i=1}^{n} f(x_i) + \sum_{p=0}^{k-1} \prod_{j=0}^{p} m(w_j) h_2(w_{p+1}) \sum_{i=1}^{n} f(x_i).
\]

When \( h_2 \left( \frac{1}{n} \right) \neq 1 \), using the inequality (Equation 4.3), we have

\[
\frac{1 - h_2 \left( \frac{1}{n} \right)}{n-1} \sum_{k=2}^{n} \sum_{i=1}^{k-1} f \left( \prod_{p=1}^{i-1} x_p^{w_{p+1} m(i)} \right) \leq \left\{ 1 - h_2 \left( \frac{1}{n} \right) \right\} \left\{ h_1(w_1) - h_2(w_1) \right\} \sum_{i=1}^{n} f(x_i)
\]

\[
+ \sum_{p=0}^{k-1} \prod_{j=0}^{p} m(w_j) h_2(w_{p+1}) \} \left\{ \sum_{i=1}^{n} f(x_i) \right\} + h_1 \left( \frac{1}{n} \right) - h_2 \left( \frac{1}{n} \right) \sum_{i=1}^{n} f(x_i)
\]

\[
\sum_{p=0}^{n-1} \prod_{j=0}^{p} m \left( \frac{1}{n} \right)^p \sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} f \left( \prod_{j=1}^{i} x_j^{2 m \left( \frac{1}{n} \right)^{i-1}} \right) \sum_{i=1}^{n} f(x_i).
\]

When \( h_2 \left( \frac{1}{n} \right) = 1 \), we obtain

\[
\frac{1}{n-1} \sum_{k=2}^{n} \sum_{i=1}^{k-1} f \left( \prod_{p=1}^{i-1} x_p^{w_{p+1} m(i)} \right) \leq h_1(w_1) + \sum_{p=0}^{k-1} \prod_{j=0}^{p} m(w_j) h_2(w_{p+1}) \sum_{i=1}^{n} f(x_i) \sum_{i=1}^{n} f(x_i)
\]

\[
\leq \left\{ h_1 \left( \frac{1}{n} \right) - 1 \right\} \sum_{i=1}^{n} f(x_i) \sum_{i=1}^{n} f \left( \prod_{j=1}^{i} x_j^{2 m \left( \frac{1}{n} \right)^{i-1}} \right).
\]

The proof of Theorem 6.2 is complete. \( \square \)

**Corollary 6.2.1** Under the conditions of Theorem 6.2, if \( w_1 = \cdots = w_k = \frac{1}{n} \),

(1) when \( 1 - h_2 \left( \frac{1}{n} \right) \neq 0 \), we have

\[
\frac{1 - h_2 \left( \frac{1}{n} \right)}{n-1} \sum_{k=2}^{n} \sum_{i=1}^{k-1} f \left( \prod_{p=1}^{i-1} x_p^{w_{p+1} m(i)} \right) \leq \left[ 1 - h_2 \left( \frac{1}{n} \right) \right] \left[ h_1 \left( \frac{1}{n} \right) - h_2 \left( \frac{1}{n} \right) \right] \sum_{i=1}^{n} f(x_i) + \sum_{p=0}^{k-1} \prod_{j=0}^{p} m \left( \frac{1}{n} \right)^p \sum_{i=1}^{n} f(x_i)
\]

\[
\times \left\{ \left[ h_1 \left( \frac{1}{n} \right) - h_2 \left( \frac{1}{n} \right) \right] \sum_{i=1}^{n} f(x_i) - \sum_{p=0}^{n-1} \prod_{j=0}^{p} m \left( \frac{1}{n} \right)^p \sum_{i=1}^{n} f(x_i) \right\};
\]
(2) when $1 - h_2\left(\frac{1}{n}\right) = 0$, we have

If $h_1(t_1) h_2(t_2) \geq h_2(t_1 t_2)$ for all $t_1, t_2 \in [0, 1]$, $h$ is sub-multiplicative, and $f$ is $(h_1, h_2, m)$-GA-concave

$$
\frac{1}{n - 1} \sum_{1 \leq j < \cdots < j_n \leq n} f\left(\prod_{j=1}^{k} x_j^{1/k}\right)
$$

$$
\leq \left[h_1\left(\frac{1}{k}\right) + h_2\left(\frac{1}{k}\right)\left(\sum_{j=1}^{k-1} m\left(\frac{1}{k}\right)^p\right)\right] \sum_{i=1}^{n} f(x_i) + \frac{h_2\left(\frac{1}{k}\right)}{\sum_{p=0}^{n-1} m\left(\frac{1}{n}\right)^p} \sum_{i=1}^{n} f\left(\prod_{j=1}^{n-1} x_j^{\frac{1}{m\left(\frac{1}{n}\right)}}\right)
$$

$$
\times \left\{h_1\left(\frac{1}{n}\right) - 1\right\} \sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f\left(\prod_{j=1}^{n-1} x_j^{\frac{1}{m\left(\frac{1}{n}\right)}}\right).
$$

on $(0, b]$, then the above inequalities are reversed.

**Corollary 6.1.2** Under the conditions of Theorem 6.2, if $w_1 = \cdots = w_k = \frac{1}{k}$ and $m(t) = 1$ for all $t \in [0, 1]$,

(1) when $1 - h_2\left(\frac{1}{n}\right) \neq 0$, we have

$$
\frac{1 - h_2\left(\frac{1}{n}\right)}{n - 1} \sum_{1 \leq j < \cdots < j_n \leq n} f\left(\prod_{j=1}^{k} x_j^{1/k}\right) \leq \left[1 - h_2\left(\frac{1}{n}\right)\right] \left[h_1\left(\frac{1}{k}\right) - h_2\left(\frac{1}{k}\right)\right]
$$

$$
\times \left[h_1\left(\frac{1}{n}\right) - 1\right] \sum_{i=1}^{n} f(x_i) - f\left(\prod_{j=1}^{n} x_j^{1/n}\right),
$$

(2) when $1 - h_2\left(\frac{1}{n}\right) = 0$, we have

If $h_1(t_1) h_2(t_2) \geq h_2(t_1 t_2)$ for all $t_1, t_2 \in [0, 1]$, $h$ is sub-multiplicative, and $f$ is $(h_1, h_2, m)$-GA-concave on $(0, b]$, then the inequalities (Equations 6.12 and 6.13) are reversed.

$$
\frac{k}{n - 1} \sum_{1 \leq j < \cdots < j_n \leq n} f\left(\prod_{j=1}^{k} x_j^{1/k}\right) \leq \left[h_1\left(\frac{1}{k}\right) + kh_2\left(\frac{1}{k}\right)\right] \sum_{i=1}^{n} f(x_i)
$$

$$
+ h_2\left(\frac{1}{k}\right) \left[\frac{1}{n} h_1\left(\frac{1}{n}\right) - 1\right] \sum_{i=1}^{n} f(x_i) - f\left(\prod_{j=1}^{n} x_j^{1/n}\right).
$$

**Corollary 6.1.3** Under the conditions of Theorem 6.2, if $w_1 = \cdots = w_k = \frac{1}{k}$ and $h_1(t) = h_2(t) = h(t)$ for all $t \in [0, 1]$,

(1) when $1 - h\left(\frac{1}{n}\right) \neq 0$, we have

$$
\frac{1 - h\left(\frac{1}{n}\right)}{n - 1} \sum_{1 \leq j < \cdots < j_n \leq n} f\left(\prod_{j=1}^{k} x_j^{1/k}\right) \leq \left(\sum_{p=0}^{n-1} m\left(\frac{1}{n}\right)^p\right) \sum_{i=1}^{n} f(x_i) - \frac{\sum_{i=1}^{n} f\left(\prod_{j=1}^{n-1} x_j^{\frac{1}{m\left(\frac{1}{n}\right)}}\right)}{\sum_{p=0}^{n-1} m\left(\frac{1}{n}\right)^p}.
$$
(2) when \(1 - h\left(\frac{1}{l}\right) = 0\), we have

\[
\frac{1}{n - 1} \binom{n-1}{k-1} h\left(1 - h\left(\frac{1}{l}\right)\right) \sum_{1 \leq r_i < \cdots < r_i \leq n} f\left(\prod_{p=1}^{k} x_{r_p}^{1/n}\right) \leq \left[1 + \sum_{p=0}^{k-1} m\left(\frac{1}{k}\right)\right]^p \prod_{i=1}^{n} f(x_i) \cdot \frac{\sum_{p=0}^{n-1} f\left(\prod_{p=1}^{n-1} x_i^{1/n}\right)}{\sum_{p=0}^{n-1} m\left(\frac{1}{p}\right)^p}.
\]

(6.15)

If \(h\) is sub-multiplicative and \(f\) is \((h_1, h_2, m)\)-GA-concave on \((0, b)\), then the inequalities (Equations 6.12 and 6.13) are reversed.

**Corollary 6.1.4** Under the conditions of Theorem 6.2, if \(w_1 = \cdots = w_n = \frac{1}{n}\) for any \(t \in [0, 1]\), \(h_2\left(\frac{1}{t}\right) \leq 1\), and \(f\) is a decreasing function on \((0, b)\), then

(1) when \(h_2\left(\frac{1}{t}\right) < 1\), we have

\[
\frac{1}{n - 1} \binom{n-1}{k-1} h\left(1 - h\left(\frac{1}{l}\right)\right) \sum_{1 \leq r_i < \cdots < r_i \leq n} f\left(\prod_{p=1}^{k} x_{r_p}^{1/n}\right) \leq \left[1 + \frac{h_1\left(\frac{1}{n}\right) - h_2\left(\frac{1}{n}\right)}{n}\right] \sum_{i=1}^{n} f(x_i) \cdot f\left(\prod_{i=1}^{n} x_i^{1/n}\right) - f\left(\prod_{i=1}^{n} x_i\right). \]

(2) when \(h_2\left(\frac{1}{t}\right) = 1\), we have

If \(h_1(t_1)h_2(t_2) \geq h_2(t_1, t_2)\) for all \(t_1, t_2 \in [0, 1]\), \(h\) is sub-multiplicative, and \(f\) is \((h_1, h_2)\)-GA-concave on \((0, b)\), then the above inequalities are reversed.
7. More remarks

Remark 5  Letting $k = 2$ in the inequality (Equation 6.9) yields the inequality (Equation 1.4).

Remark 6  Letting $h_1(t) = h_2(t) = h(t)$ in the first inequality in Corollary 6.2.3 for all $t \in (0, 1)$ yields the inequality (39) in Xi et al. (2014b).