Research Article

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An inequality involving the gamma and digamma functions

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Abstract: In the paper, the authors establish an inequality involving the gamma and digamma functions and apply it to prove the negativity and monotonicity of a function involving the gamma and digamma functions.

Keywords: Gamma function, psi function, polygamma function, inequality, negativity, monotonicity, application

MSC 2010: Primary 33B15; secondary 26A48, 26D07

1 Introduction

It is common knowledge that the classical Euler gamma function \( \Gamma(x) \) may be defined for \( x > 0 \) by

\[
\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt.
\]

The logarithmic derivative of \( \Gamma(x) \), denoted by

\[
\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},
\]

is called the psi or digamma function, and the \( \psi^{(k)}(x) \) for \( k \in \mathbb{N} \) are called the polygamma functions. It is well known that these functions are fundamental and that they have much extensive applications in mathematical sciences.

The aim of this paper is to establish an inequality involving the gamma and digamma functions. The result is stated as the following theorem.

**Theorem 1.1.** For \( t \in (0, \infty) \), we have

\[
\frac{1 + 2t}{2t^2} \left[ \ln \Gamma \left( \frac{t}{1 + 2t} \right) - \ln \Gamma(t) \right] < 1 - \psi(t).
\] (1.1)

As applications of Theorem 1.1, the following negativity and monotonicity are obtained.

**Theorem 1.2.** For \( y \in (-1, -\frac{1}{2}) \), the function

\[
x \psi(x + y + 1) - \ln \Gamma(x + y + 1) + \ln \Gamma(y + 1) - \frac{x^2}{2(y + 1)(x + y + 1)}
\] (1.2)

is negative and decreasing with respect to \( x \in [-\frac{2(y+1)^2}{1+2y}, \infty) \).

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Remark 1.3. Inequality (1.1) can be reformulated as

\[
\frac{\ln \Gamma\left(\frac{t}{t+2}\right)}{\frac{t}{t+2}} - \ln \Gamma(t) < 1 - \psi(t)
\]

for \( t \in (0, \infty) \).

Remark 1.4. The negativity of the function (1.2) is equivalent to

\[
\left[ \frac{\Gamma(x+t)}{\Gamma(t)} \right]^{\frac{1}{t}} > \exp\left( \psi(x+t) - \frac{x}{2t(x+t)} \right)
\]

for \( t \in (0, \frac{1}{2}) \) and \( x \in [-\frac{2x^2}{2t+1}, \infty) \).

## 2 Lemmas

In order to prove our main results, the following lemmas are needed.

**Lemma 2.1** ([13, p. 305]). For \( x > 0 \), we have

\[
\frac{1}{2x} - \frac{1}{12x^2} < \psi(x+1) - \ln x < \frac{1}{2x}.
\]

(2.1)

**Lemma 2.2** ([3, Lemma 1] and [10, Theorem 1]). The inequality

\[
e^\psi(L(a,b)) < \left[ \frac{\Gamma(a)}{\Gamma(b)} \right]^{\frac{1}{b}}
\]

(2.2)

is valid for positive numbers \( a \) and \( b \) with \( a \neq b \), where

\[
L(a, b) = \frac{b - a}{\ln b - \ln a}
\]

(2.3)

stands for the logarithmic mean.

**Lemma 2.3** ([19, Theorem 1]). For real numbers \( s > 0 \) and \( t > 0 \) with \( s \neq t \) and an integer \( i \geq 0 \), the inequality

\[
(-1)^i \psi^{(i)}(L_p(s,t)) \leq \frac{(-1)^i}{t - s} \int_s^t \psi^{(i)}(u) \, du \leq (-1)^i \psi^{(i)}(L_q(s,t))
\]

(2.4)

holds if \( p \leq -i - 1 \) and \( q \geq -i \), where \( L_p(a,b) \) is the generalized logarithmic mean of order \( p \in \mathbb{R} \) for positive numbers \( a \) and \( b \) with \( a \neq b \),

\[
L_p(a,b) = \begin{cases} 
\left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}, & p \neq -1, 0, \\
\frac{b - a}{\ln b - \ln a}, & p = -1, \\
\frac{1}{e} \left( \frac{b^p}{a^p} \right)^{\frac{1}{p}}, & p = 0.
\end{cases}
\]

**Lemma 2.4.** For \( x \in (0, \infty) \) and \( k \in \mathbb{N} \), we have

\[
\ln\left(x + \frac{1}{2}\right) - \frac{1}{x} < \psi(x) < \ln(x + 1) - \frac{1}{x}
\]

(2.5)

and

\[
\frac{(k-1)!}{(x+1)^k} + \frac{k!}{x^{k+1}} < (-1)^{k+1} \psi^{(k)}(x) < \frac{(k-1)!}{(x + \frac{1}{2})^k} + \frac{k!}{x^{k+1}}.
\]

(2.6)
Proof. In [6, Theorem 1], the following necessary and sufficient conditions are obtained: For real numbers $\alpha \neq 0$ and $\beta$, the function
\[
g_{\alpha, \beta}(x) = \left[ \frac{e^x \Gamma(x+1)}{(x+\beta)^{\alpha}} \right]^{\alpha}, \quad x \in (\max\{0,-\beta\}, \infty)
\]
is logarithmically completely monotonic if and only if either $\alpha > 0$ and $\beta \geq 1$ or $\alpha < 0$ and $\beta \leq \frac{1}{2}$. Further considering the fact in [1, p.98] that a completely monotonic function which is non-identically zero cannot vanish at any point on $(0, \infty)$ gives
\[
(-1)^k [\ln g_{\alpha, \beta}(x)]^{(k)} = (-1)^k \alpha [x + \ln \Gamma(x) + \ln x - (x + \beta) \ln(x + \beta)]^{(k)} > 0
\]
for $k \in \mathbb{N}$ and $x \in (0, \infty)$ if and only if either $\alpha > 0$ and $\beta \geq 1$ or $\alpha < 0$ and $\beta \leq \frac{1}{2}$. As a result, from straightforward calculation and standard arrangement, inequalities (2.5) and (2.6) follow.

Lemma 2.5 ([8, p. 296] and [9, p. 274, (3.6.19)]). For $x > 0$, we have
\[
\ln \left( 1 + \frac{1}{x} \right) < \frac{2}{2x+1} \left[ 1 + \frac{1}{12x} - \frac{1}{12(2x+1)} \right]. \tag{2.7}
\]

Lemma 2.6 ([4, p. 107, Lemma 3]). For $x \in (0, \infty)$ and $k \in \mathbb{N}$, we have
\[
\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}
\]
and
\[
\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^k \psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}}. \tag{2.8}
\]

Remark 2.7. It is noted that the left-hand side of the double inequality (2.4) for $i = 0$ and $p = -1$ is just inequality (2.2). For more information about inequalities (2.2) and (2.4), we refer to [11, 12, 16–18, 20, 21] and related references therein.

Remark 2.8. In [2, Corollary 3] and [5, Theorem 1], it was proved that the double inequality
\[
\ln \left( x + \frac{1}{2} \right) - \frac{1}{x} < \psi(x) < \ln(x + e^{-x}) - \frac{1}{x} \tag{2.9}
\]
holds on $(0, \infty)$. In [5, Theorem 1], it was also shown that the scalars $\frac{1}{2}$ and $e^{-x} = 0.56 \ldots$ in (2.9) are the best possible. It is obvious that inequality (2.9) refines and sharpens (2.5).

Remark 2.9. In [7], inequality (2.6) was refined and sharpened.

3 Proofs of theorems

Now we are in a position to prove our theorems.

Proof of Theorem 1.1. It is clear that
\[
\frac{1 + 2t}{2t^2} \left[ \ln \Gamma\left( \frac{t}{1 + 2t} \right) - \ln \Gamma(t) \right] = \frac{1 + 2t}{2t^2} \int_{t/(1+2t)}^{t} \psi(u) \, du,
\]
so the required inequality (1.1) can be rewritten as
\[
\frac{1 + 2t}{2t^2} \int_{t/(1+2t)}^{t} \psi(u) \, du - \psi(t) + 1 \triangleq p(t) > 0, \quad t > 0.
\]
Using the equality
\[
\psi(x + 1) = \psi(x) + \frac{1}{x}
\]
for $x > 0$, it is easily seen that inequality (2.1) is equivalent to
\[
\ln x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) < \ln x - \frac{1}{2x}, \quad x > 0. \tag{3.1}
\]
From (3.1), it follows that
\[
p(t) > \frac{1 + 2t}{2t^2} \int_{t/(1+2t)}^t \left( \ln u - \frac{1}{2u} - \frac{1}{12u^2} \right) du - \left( \ln t - \frac{1}{2t} \right) + 1 = \frac{4t - 3 \ln(2t + 1) - 1}{12t^2} \pm \frac{q(t)}{12t^2}.
\]

Since \( q'(t) = \frac{2^q(t-1)}{2t+1} \), the function \( q(t) \) is increasing for \( t > \frac{1}{2} \). By \( q(\frac{2}{7}) = 0.002 \ldots > 0 \), it is easy to see that inequality (1.1) holds for \( t \geq \frac{2}{7} = 1.1428 \ldots 

Letting \( a = t \) and \( b = \frac{t}{1+2t} \) in (2.2) and (2.3) gives
\[
\frac{\ln \Gamma(t/(1+2t)) - \ln \Gamma(t)}{t/(1+2t) - t} > \psi\left(1 + \frac{2t^2}{(1+2t) \ln(1+2t)}\right) = \int_{2t/(1+2t) \ln(1+2t)}^t \psi'(u) du < 1 \tag{3.2}
\]

for \( t > 0 \), it is sufficient to show that
\[
\psi(t) - \psi\left(1 + \frac{2t^2}{(1+2t) \ln(1+2t)}\right) = \int_{2t/(1+2t) \ln(1+2t)}^t \psi'(u) du < 1
\]

holds on \( (0, \frac{2}{7}) \).

Taking \( s = \frac{2t^2}{(1+2t) \ln(1+2t)} \), \( i = 1 \) and \( p = -2 \) in the left-hand side of inequality (2.4) leads to
\[
\int_{2t/(1+2t) \ln(1+2t)}^t \psi'(u) du \leq \left( t - \frac{2t^2}{(2t+1) \ln(2t+1)} \right) \psi\left(\left( \frac{2t^3}{(2t+1) \ln(2t+1)} \right)^{\frac{1}{2}}\right).
\]

Combining this with (3.2) reveals that it suffices to prove
\[
\psi\left(\left( \frac{2t^3}{(2t+1) \ln(2t+1)} \right)^{\frac{1}{2}}\right) \leq \frac{2t^3}{(2t+1) \ln(2t+1)} + \frac{1}{\sqrt{2t^3/(2t+1) \ln(2t+1) + 1/2}} \leq \frac{2t^3}{(2t+1) \ln(2t+1) - 2t} \tag{3.3}
\]

for \( t \in (0, \frac{2}{7}) \).

The right-hand side of the double inequality (2.6) for \( k = 1 \) results in
\[
\psi\left(\left( \frac{2t^3}{(2t+1) \ln(2t+1)} \right)^{\frac{1}{2}}\right) \leq \frac{2t^3}{(2t+1) \ln(2t+1)} + \frac{1}{\sqrt{2t^3/(2t+1) \ln(2t+1) + 1/2}} \leq \frac{2t^3}{(2t+1) \ln(2t+1) - 2t} \tag{3.4}
\]

for \( t \in (0, \frac{2}{7}) \).

Inequality (2.7) can be rewritten as
\[
\ln(1+t) < \frac{t(t^2 + 12t + 12)}{6(t+1)(t+2)}, \quad t > 0.
\]

Therefore, to verify (3.4), it is sufficient to prove
\[
\frac{(2t+1) \ln(2t+1)}{2t^3} \cdot \frac{2t(4t^2 + 24t + 12)}{6(2t+1)(2t+2)} + \frac{1}{\sqrt{2t^3 - 6(2t+1)(2t+2)}} + \frac{1}{2} \leq \frac{1}{1 - \frac{2t^2}{2t^2 + 6(2t+1)(2t+2)}} \tag{3.5}
\]

for \( t \in (0, \frac{2}{7}) \), which can be simplified as
\[
\frac{t^2 + 6t + 3}{3t^2(t+1)} + \frac{2}{1 + \sqrt{12t^2(t+1)/(t^2 + 6t + 3)}} \leq \frac{t^2 + 6t + 3}{t^2(t+3)} \tag{3.6}
\]
Since \( q(t) \) is increasing on \((0, \infty)\) with \( q(0) = -3 \) and \( q(1) = 14 \), the function \( q(t) \) has a unique zero \( t_0 \in (0, 1) \). From \( q(\frac{1}{2}) = -\frac{7}{16} \), we can locate more accurately that \( t_0 \in (\frac{1}{2}, 1) \). When \( 0 < t \leq t_0 \), the function \( q(t) \) is non-positive, so inequality (3.6) is clearly valid. When \( t \geq t_0 \), the function \( q(t) \) is non-negative, so squaring both sides of (3.6) and simplifying gives

\[
h(t) = 9t^6 + 54t^5 + 55t^4 - 60t^3 - 93t^2 - 18t + 9 \leq 0.
\]

Direct differentiation yields

\[
\begin{align*}
h'(t) &= 54t^5 + 270t^4 + 220t^3 - 180t^2 - 186t - 18, \\
h''(t) &= 270t^4 + 1080t^3 + 660t^2 - 360t - 186, \\
h^{(3)}(t) &= 1080t^3 + 3240t^2 + 1320t - 360.
\end{align*}
\]

It is clear that the function \( h^{(3)}(t) \) is increasing with \( \lim_{t \to -\infty} h^{(3)}(t) = \infty \) and \( h^{(3)}(0) = -360 \), so the function \( h^{(3)}(t) \) has a unique zero which is the unique minimum point of the function \( h''(t) \). Since \( h''(0) = -186 \) and \( \lim_{t \to -\infty} h''(t) = \infty \), the function \( h''(t) \) has a unique zero which is the unique minimum point of the function \( h'(t) \). From \( h'(0) = -186 \) and \( \lim_{t \to -\infty} h'(t) = \infty \), we conclude that the function \( h'(t) \) has a unique zero which is the unique minimum point of the function \( h(t) \) on \((0, \infty)\). Due to \( h(0) = 9 \), \( h(\frac{1}{2}) = -\frac{700}{81} \), \( h(\frac{8}{7}) = -\frac{604759}{177649} \) and \( \lim_{t \to -\infty} h(t) = \infty \), it is not difficult to see that the function \( h(t) < 0 \) on \( (\frac{1}{7}, \frac{8}{7}) \). As a result, inequalities (3.6), and so (3.5), holds on \((0, \frac{1}{7})\). The proof of Theorem 1.1 is complete. \( \square \)

**Proof of Theorem 1.2.** Denote the function (1.2) by \( q(x, y) \). Differentiating and using the right-hand side of inequality (2.8) yields

\[
\frac{\partial q(x, y)}{\partial x} = x \left[ \psi'(x + y + 1) - \frac{(x + 2y + 2)}{2(y + 1)(x + y + 1)^2} \right] \\
< x \left[ \frac{1}{x + y + 1} + \frac{1}{(x + y + 1)^2} - \frac{(x + 2y + 2)}{2(y + 1)(x + y + 1)^2} \right] \\
= x \left[ \psi(x + 1 + 2y) + 2(y + 1)^2 \right] \\
\leq x \frac{(x + 1 + 2y)(x + y + 1)^2}{2(y + 1)(x + y + 1)^2}.
\]

the function \( \frac{\partial q(x, y)}{\partial x} \) is negative for \( (x, y) \in [\frac{2(y + 1)^2}{1 + 2y}, \infty) \times (\frac{1}{2}, \infty) \), so the function \( q(x, y) \) is decreasing with \( x \in [\frac{2(y + 1)^2}{1 + 2y}, \infty) \) for \( y \in (-1, \frac{1}{2}) \). Furthermore, from

\[
\begin{align*}
q\left(\frac{2(y + 1)^2}{1 + 2y}, y\right) &= \frac{2(y + 1)^2}{2y + 1} \left[ 1 - \psi\left(\frac{y + 1 + 2y}{2y + 1}\right) \right] - \ln \Gamma\left(\frac{y + 1 + 2y}{2y + 1}\right) + \ln \Gamma(y + 1) \\
&= \frac{2(y + 1)^2}{2y + 1} \left[ 1 - \psi\left(\frac{y + 1 + 2y}{2y + 1}\right) \right] + \ln \Gamma\left(-\frac{(y + 1)(2y + 1)}{(y + 1)(2y + 1)}\right)
\end{align*}
\]

and inequality (1.1), it follows that the function \( q\left(\frac{2(y + 1)^2}{1 + 2y}, y\right) \) is negative for \( y \in (-1, \frac{1}{2}) \), and so the function \( q(x, y) \) is negative for \( x \in [\frac{2(y + 1)^2}{1 + 2y}, \infty) \) and \( y \in (-1, \frac{1}{2}) \). The proof of Theorem 1.2 is complete. \( \square \)

**Remark 3.1.** This paper is a slightly revised version of the preprint [15] and a part of the preprint [14]. Another part was formally published in [22].

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**References**


