Online Appendix to "Growth and Ideas under Uncertainty"

Abstract
Stochastic growth models are generally impossible to solve in closed form. Despite their intractability, several recent studies have found closed-form solutions to AK or Uzawa-Lucas growth models. Following their lead, I analyze the stochastic Romer model in which endogenous technological progress follows a combination of a controlled diffusion process and many Poisson jump processes. Finding its explicit solution, I demonstrate that larger technology shocks deteriorate the welfare of households and slow down economic growth. Simple simulation for Japan suggests the importance of stochastic elements in accounting for technological progress. Moreover, in another specification, I show that higher uncertainty induces more consumption.

Keywords: Technology; Welfare; Stochastic Growth; Analytical Solution

JEL: C60, O33, O41

1. Introduction
Stochastic growth models have played a central role in macroeconomics. Since the seminal work of Brock and Mirman (1972), they have been extensively used to analyze the role of uncertainty in the growth process, as well as in business cycle research. They have, however, one shortcoming: they are generally impossible to solve in closed form (Merton, 1975). A survey of Wälde (2011) reveals that, to obtain the explicit solution to stochastic growth models, production technology should be, in most cases, confined to the AK type. Use of AK models are, however, widely criticized. For example, they do not make an explicit distinction between technological progress and capital accumulation (Aghion and Howitt, 2009, p.66). Moreover, Solow (2005) is persuasively dismissive of AK models by saying that they mislead guides to policy. Therefore, the scope and applicability of stochastic growth models have largely been limited.

These being said, recent several studies have succeeded in finding closed-form solutions to stochastic growth models beyond AK models. For example, imposing one or two parameter restriction(s), Bucci et al. (2011) and Hiraguchi (2013) obtain the explicit solution to the stochastic two-sector optimal growth model of Uzawa (1965) and Lucas (1988) in which exogenous technological progress is driven by a geometric Brownian motion process. In a similar vein, Marsiglio and La Torre (2012a, 2012b) find the closed-form solution to the stochastic Uzawa-Lucas model in which exogenous population dynamics is governed by a geometric Brownian motion process, with two parameter restrictions. Imposing one parameter restriction, Hiraguchi (2014) analytically solves the stochastic Ramsey model with leisure in which exogenous technological progress follows a combination of a geometric Brownian...
motion process and many Poisson jump processes. Menoncin and Nembrini (2018) analytically solve stochastic growth models with HARA (Hyperbolic Absolute Risk Aversion) preferences in which exogenous technological progress is driven by a mixture of a Brownian motion process and a Poisson jump process, at the cost of (essentially) three restrictions.

They all are valuable contributions, as they demonstrate that closed-form solutions to stochastic growth models are possible without confining production technology to the AK type that draws several criticisms. Moreover, their studies indicate great possibilities in stochastic growth models to be applied more widely to a richer class of models such as the Uzawa-Lucas two-sector growth model. At the same time, studies cited above hold strong possibilities to be explored further. Specifically, there are three points that need a fuller inquiry.

First, they all assume exogenous stochastic dynamics for technological progress (or population growth). This means that dynamics is not well explained within a model. For example, in Bucci et al. (2011) and Hiraguchi (2013), technological progress is assumed to follow a geometric Brownian motion process in the Uzawa-Lucas model. It is, however, a human-capital-based endogenous growth model. As such, they are unable to examine the impacts of higher uncertainty on technological progress. Given that the Uzawa-Lucas growth model is not the only model of endogenous growth, I instead analyze the stochastic Romer (1990) model in which endogenous technological progress follows a combination of a controlled diffusion process and many Poisson jump processes. Note that no study has found the closed-form solution to the stochastic Romer (1990) model. As technological change is endogenous in the Romer (1990) model, if we can find the closed-form solution, it becomes possible to analytically characterize the impacts of higher uncertainty on technological progress.

Second, related to the first, there is a good reason to incorporate the stochastic elements into the Romer (1990) model — the outcome of R&D is not deterministic. For example, Solow (1994, p.52) says that ”...in the frequency with which research projects end up by finding something that was not even contemplated when the initial decisions were made...a second difficulty...is the large uncertainty surrounding many research projects.” In other words, not all R&D is successful. If we explicitly take uncertainty inherent in the R&D process into account, will rates of technological progress accelerate or slow down? The answer is indeed not easy to pin down, as there are two possible outcomes. On the one hand, higher uncertainty may reduce the willingness of firms to invest in R&D, thereby slowing down technological progress. On the other, faced with the more uncertain future, firms may appear more willing to innovate, hence stimulating R&D (Bloom, 2014). I analytically show that, at least in the stochastic Romer (1990) model, the former is correct; that is, larger technology shocks slow down the rate of technological progress by allocating less of the total stock of human capital in an economy to the R&D sector. This prediction is consistent with the findings of empirical studies that demonstrate a negative relationship between uncertainty and growth, such as Ramey and Ramey (1995) and Norrbin and Yigit (2005).

Third, they all do not cover the welfare analysis. Nevertheless, as Turnovsky (1997, 2000) show, when stochastic growth models can be solved in closed form, they are remarkably suitable for inspecting the nexus between uncertainty and welfare. Consequently, I use
the explicit solution to examine the impact of larger technology shocks (and later, shocks to capital) on the welfare of agents. Tsuboi (2018) analyzes this in the context of the stochastic Uzawa-Lucas model in which the accumulation of human capital is driven by stochastic processes, and shows that higher uncertainty about human capital accumulation deteriorate welfare. In this paper, I show that larger technology shocks and shocks to physical capital are both welfare-reducing, in the context of the stochastic Romer (1990) model. Furthermore, I analyze the impacts of Poisson uncertainty on welfare (and growth). In particular, Bucci et al. (2011) and Hiraguchi (2013) only explore Brownian uncertainty. Among studies cited above, it is only Tsuboi (2018) that examines the effects of Poisson uncertainty on welfare (and growth). In this regard, this paper also differs from previous studies: that it uncovers the relationship between Poisson uncertainty and welfare (and growth).

To sum up, the purpose of this paper is finding the closed-form solution to the stochastic Romer (1990) model in which endogenous technological progress is driven by a mixture of a controlled diffusion process and many Poisson jump processes and, based on the explicit solution, to analyze the impacts of higher Brownian and Poisson uncertainty about technological progress on its growth path and welfare. The paper is organized as follows. Sect. 2 sets up and solves the stochastic Romer (1990) model. Sect. 3 considers another specification. Concluding remarks appear in Sect. 4.

2. The Model

In this section, I construct a stochastic version of the Romer (1990) model. The economy has three sectors. First, the research sector uses human capital and the existing stock of knowledge to produce new knowledge (designs for new producer durables). Second, an intermediate-goods sector uses the designs from the research sector together with forgone output to produce the large number of product durables available for use in final-goods production at any time. Third, a final-goods sector uses labor, human capital, and the set of producer durables available to produce final output. Output can be either consumed or saved as new capital. To keep the dynamic analysis simple, as in Romer (1990), I assume that the population and the supply of labor are both constant.

2.1. Preferences and Capital Accumulation

The Romer (1990) model is so stylized in the literature that, for our purpose, I will not go over the model step by step. Throughout the paper, I simplify the notation by suppressing time index when this causes no confusion. Consider a closed economy in continuous time running to an infinite horizon. The economy is inhabited by a larger number of households. I assume that they all are identical, so that the economy admits a representative household with CRRA (Constant Relative Risk Aversion) preferences

$$E \int_0^\infty e^{-\rho t} \frac{C^{1-\phi} - 1}{1 - \phi} dt, \quad (1)$$

I will not discuss (strong) scale effects pertaining to standard endogenous growth models. See Jones (1995, 2005) for an excellent discussion on them.
with the coefficient of relative risk aversion given by \( \phi > 0 \). Time-separable utility functions imply that the inverse of the elasticity of intertemporal substitution and the coefficient of relative risk aversion are identical (Acemoglu, 2009). When future consumption is uncertain, a larger \( \phi \) makes future utility gain smaller, raising the value of additional future consumption. \( \mathbb{E} \) is the expectation operator with respect to the information set available to a representative household. \( \rho > 0 \) denotes the subjective discount rate of the household, that is, the rate at which utility is discounted.

The total amount of human capital available to the economy \( H \) is constant. It is open to the economy to allocate this given stock between the production of output \( H_Y \) and the production of new varieties of capital \( H_A \), that is, \( H_Y + H_A = H \). Supposing that there is a continuum of capital (intermediate) goods running 0 to \( A \), the technology for producing final output \( Y \) is

\[
Y = H_Y^\alpha L^\beta \left( \int_0^A x(i)di \right)^{1-\alpha-\beta},
\]

where \( \alpha \in (0, 1), \beta \in (0, 1) \), and \( L \) is the constant amount of raw labor available. \( x(i) \) is the amount of the \( i \)th capital good available for production. As \( i \in [0, A] \), we have a density of capital goods \( x(i) \). Moreover, since it takes \( \eta \) units of forgone consumption to create one unit of any type of capital good, it takes \( \eta x(i) \) units of the resource to produce \( x(i) \) units of the \( i \)th capital good for all \( i \). As such, the physical capital stock \( K \) is related to the durable goods that are actually used in production by the rule

\[
K = \eta \int_0^A x(i)di.
\]

Because of the symmetry, all the durable goods available are supplied at the same level, say \( \bar{x} \). Note that, despite diminishing returns to each of capital goods, we can equalize the marginal products of the \( x(i) \) only by equalizing the \( x(i) \). This implies \( K = \eta A \bar{x} \). As a consequence, (2) can be rewritten as

\[
Y = \eta^{\alpha+\beta-1} A^{\alpha+\beta} H_Y^\alpha L^\beta K^{1-\alpha-\beta},
\]

which can be used to describe the law of motion for physical capital \( K \):

\[
dK = \underbrace{\eta^{\alpha+\beta-1} A^{\alpha+\beta} H_Y^\alpha L^\beta K^{1-\alpha-\beta}}_{\equiv Y} dt - Cdt - \delta Kdt,
\]

where \( \delta \in (0, 1) \) is the depreciation rate of physical capital.\footnote{In the original Romer (1990) model, \( \delta = 0 \). I include it for a bit of generality.}

2.2. Stochastic Technological Progress

The key equation in this paper is the law of motion for \( A \), a count of the number of designs or the entire stock of knowledge. As knowledge is a nonrival input, all researchers
can take advantage of $A$ at the same time. This is the quintessence of the Romer (1990) model: technological progress is endogenous. The innovation here is that, following Hiraguchi (2014), I assume that $A$ follows a combination of a Brownian motion process and many Poisson jump processes:

$$dA = \mu HA dt + \sigma_a A dz_a + \sum_{j=1}^{N} \beta_j A dq_j,$$

where $\mu$ is a productivity parameter and $H_A$ has the clear interpretation of total human capital employed in research. $\sigma_a > 0$ is the diffusion coefficient of technology (if $\sigma_a = 0$, we would recover the deterministic limit). $dz_a$ is the increment of a Brownian (or Wiener) process such that the mean $\mathbb{E}(dz_a) = 0$ and variance $\mathbb{V}(dz_a) = dt$. In other words, changes in a Brownian motion process over any finite interval of time are normally distributed, with a variance that increases linearly with the time interval. As the Brownian motion process is nonstationary, its variance will go to infinity over the long run. In addition, there are $N$ independent Poisson jump processes $q_j$ with the arrival rate $\lambda_j$ and a jump of size $\beta_j > -1$.

I further assume the initial stock of knowledge $A(0) = A_0 > 0$ so that $A(t) > 0$ for all $t$ with probability 1, as it must.

Eq. (5) crucially differs from that of previous studies, as it is the controlled diffusion process (that is, the drift term contains one of control variables $H_A$). For example, Bucci et al. (2011) and Hiraguchi (2013) assume that technological progress in the stochastic Uzawa-Lucas model is driven by an exogenous geometric Brownian motion process only (hence no Poisson jump). Hiraguchi (2014) assumes that technological progress in the stochastic Ramsey model follows a mixture of an exogenous geometric Brownian motion process and many Poisson jump processes. As such, they cannot examine how the expected growth rate of technology is determined explicitly within their model. In contrast, because Eq. (5) includes one of control variables $H_A$, I can analyze, among many, the impacts of human capital allocation between the research sector and a final-goods sector on the expected growth rate (and on welfare, both of which are the central theme of this paper).

2.3. Optimization

The social planning problem for this economy is to maximize the expected utility (1) subject to the law of motion for physical capital (4), the stochastic differential equation for technology (5), and the resource constraint $H_Y + H_A = H$. To solve this stochastic optimization problem in continuous time, I use the Hamilton-Jacobi-Bellman (HJB) equation. With $J(K, A)$ being the value function (or indirect utility function), the HJB equation is

$$J_t + \max_{\beta_1, \ldots, \beta_N} \left( \frac{1}{2} \sigma_a^2 A^2 \beta_1^2 + H_A \mu A \beta_1 - H_A \beta_1^2 \right) = 0.$$

The solution to Eq. (5) is (net of jumps)

$$A(t) = A(0)e^{\left(\mu H_A - \frac{\sigma_a^2}{2}\right)t + \sigma_a z_a(t)}.$$ 

You can see that, though $z_a$ is unbounded, $A(t) > 0$ for all $t$ with probability 1 if the initial stock $A(0)$ is positive. Put differently, $A(t)$ is bounded from below, as it must. See Chang (2004) for details.
\[ \rho J(K, A) = \frac{C^{1-\phi} - 1}{1 - \phi} + \frac{E}{dt} \left( J_K dK + J_A dA + \frac{J_{AA}}{2} (dA)^2 + \sum_{j=1}^{N} (J(K, (1 + \beta_j)A) - J(K, A)) \right) \]

\[ = \frac{C^{1-\phi}}{1 - \phi} - \frac{1}{1 - \phi} + J_K Y - J_K C - J_K \delta K + \mu H_A A J_A + \frac{J_{AA} \sigma^2 A^2}{2} \]

\[ + \sum_{j=1}^{N} \lambda_j (J(K, (1 + \beta_j)A) - J(K, A)), \]

where \( J_K \equiv \partial J/\partial K \), \( J_A \equiv \partial J/\partial A \), and \( J_{AA} \equiv \partial^2 J/\partial A^2 \). As control variables are \( C \) and \( H_A \), the relevant first-order conditions are

\[ C = J_K^{\frac{1}{\phi}}, \] (7)

and

\[ H_A = H - \left( \frac{\alpha \eta^{\alpha + \beta - 1} L^\beta J_K K^{1-\alpha-\beta}}{\mu J_A A^{1-\alpha-\beta}} \right)^{\frac{1}{1 - \alpha}}. \] (8)

Substituting these first-order conditions (7) and (8) back into the HJB equation (6), we arrive at the maximized HJB equation:

\[ \rho J(K, A) = \frac{\phi}{1 - \phi} J_K^{\frac{\phi-1}{\phi}} - \frac{1}{1 - \phi} - J_K \delta K + (1 - \alpha) J_K^{\frac{1}{1 - \alpha}} \eta^{\alpha + \beta - 1} A^{\frac{\beta}{1 - \alpha}} L^\beta K^{\frac{\beta}{1 - \alpha}} J_A A^{\alpha - \beta} \left( \frac{\alpha}{\mu} \right)^{\frac{1}{1 - \alpha}} \]

\[ + \mu J_A AH + \frac{J_{AA} \sigma^2 A^2}{2} + \sum_{j=1}^{N} \lambda_j (J(K, (1 + \beta_j)A) - J(K, A)). \] (9)

Note that this is a partial differential equation. In general, solving it in closed form is impossible, and instead it must be solved by numerical methods such as the finite-difference method (Farlow, 1993). However, the closed-form solution gets possible with one parameter restriction. It can be summarized as follows.

**Theorem 1.** If we impose the following parameter restriction originally proposed by Xie (1991) and extensively used in the literature since then,

\[ \phi = 1 - \alpha - \beta, \] (10)

we can find the closed-form representation of the value function \( J(K, A) \) that satisfies the transversality condition (TVC) of the form
\[ J(K, A) = X K^{\alpha+\beta} + Y A^{\alpha+\beta} + Z, \]  

where

\[ X \equiv \frac{1}{\alpha + \beta} \left( \frac{1 - \alpha - \beta}{\rho + \delta(\alpha + \beta)} \right)^{1-\alpha-\beta}, \]  

\[ Y \equiv \left( \frac{\alpha}{\mu} \right)^{\alpha} \frac{XL^{\beta}}{\eta^{1-\alpha-\beta}} \left( \frac{(1 - \alpha)(\alpha + \beta)}{\rho - \mu H(\alpha + \beta) + \frac{\sigma^2}{2} (\alpha + \beta)(1 - \alpha - \beta) - \sum_{j=1}^{N} \lambda_j ((1 + \beta_j)^{\alpha+\beta} - 1)} \right)^{1-\alpha}, \]  

and

\[ Z \equiv -\frac{1}{\rho(\alpha + \beta)}. \]  

Moreover, we can also find the explicit expressions for two control variables

\[ C = ((\alpha + \beta)X)^{-\frac{1}{\alpha+\beta}} K \]
\[ = \frac{\rho + \delta(\alpha + \beta)}{1 - \alpha - \beta} K, \]  

\[ H_A = H - \left( \frac{\alpha L^{\beta} X}{\mu \eta^{1-\alpha-\beta} Y} \right)^{1-\alpha} \]
\[ = H - \frac{\alpha (\rho - \mu H(\alpha + \beta) + \frac{\sigma^2}{2} (\alpha + \beta)(1 - \alpha - \beta) - \sum_{j=1}^{N} \lambda_j ((1 + \beta_j)^{\alpha+\beta} - 1))}{\mu(1 - \alpha)(\alpha + \beta)}. \]  

Proof. See Appendix A.

2.4. Remarks

I in turn comment on main points in Theorem 1.
2.4.1. Parameter Restriction, Value Function, and Consumption

The parameter restriction (10) says that the risk aversion parameter equals the physical capital share of income. Whether it holds true in practice is still open debate, because the estimate of $\phi$ is a task of great difficulty. Nonetheless, some studies (such as Smith, 2007) report that $\phi$ should be smaller than 1 as here. Moreover, this restriction has been widely used by a number of authors, in order to obtain the closed-form solution to their model. Rebelo and Xie (1999), Smith (2007), Bucci et al. (2011), Marsiglio and Torre (2012a, b), Hiraguchi (2013, 2014), Tsuboi (2018), and Menoncin and Nembrini (2018) all use the restriction (10) to generate the insights that cannot be appreciated without the explicit solution. Following these studies, I also use (10).

It allows us to find out the closed-form representation of the value function (11) and two control variables (15) and (16). As in Turnovsky (1997, 2000) and Tsuboi (2018), we can use the value function (11) for the welfare analysis. Since (11) is represented in closed form, below, the welfare analysis can be performed simply by differentiating $J(K, A)$ with respect to the parameter of interest. Eq. (15) says that the consumption-capital ratio is constant. Though it is a little at odds that the optimal level of consumption $C$ is independent of the stock of knowledge $A$, this property is consistent with earlier findings of Smith (2007), Wälde (2011), and Hiraguchi (2013, 2014). Eq. (16) shows that the allocation of human capital to the research sector $H_A$ is dependent on a variety of parameters, especially on the technology shock $\sigma_a$ and the associated Poisson uncertainty terms. Focusing on these, we can analyze how they affect welfare and expected growth rates of technological progress.

2.4.2. Welfare

To begin with, we have

$$\frac{\partial J(K, A)}{\partial \sigma_a} < 0,$$

that is, the larger technology shock deteriorates the welfare of agents. This is illustrated in the lower left panel of Fig. 4. To see why, recall from (10) that the allocation of human capital to the research $H_A$ is decreasing in $\sigma_a$, as the upper left panel of Fig. 4 shows. Notice that, this means more of human capital is allocated to the production of final goods in parallel, as the upper right panel of Fig. 4 displays. As $H_Y$ increases, output $Y$ also increases. In sum, the larger technology shock is welfare-reducing because they allocate less of the total stock of human capital to the research sector, hence slowing down technological progress. Another way to see this is to realize that the closed-form representation of the value function $J$ is increasing in technology $A$. From Eq. (5), it is clear that a decrease in $H_A$ due to higher $\sigma_a$ lowers technological progress. This is equivalent to saying that the contribution of $A$ to $J$ is reduced by higher $\sigma_a$. Note that this mechanism also works through the constant $Y$ in which $J$ is also increasing.

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4Following Mankiw et al. (1992), I set $\alpha = \beta = 1/3$, $\delta = 0.02$, and $\mu = 0.03$. $\rho = 0.05$ is standard (typically $\rho \in [0.02, 0.05]$) in the literature. $\eta = 1$ is purely for simplicity.
The mechanism through which Poisson uncertainty affects welfare is so similar that I keep the exposition minimum in what follows. First, we have \( \partial J / \partial \beta_j > 0 \), that is, a jump of bigger size of technology is unambiguously welfare-improving. This can be understood by noting that \( \partial H_A / \partial \beta_j > 0 \). Put differently, a jump of bigger size cause more of human capital to be allocated to the research sector, hence stimulating R&D and improving welfare. Second, the effects of higher arrival rates \( \lambda_j \) is ambiguous. We have \( \partial J / \partial \lambda_j > 0 \) when \( \beta_j > 0 \) and \( \partial J / \partial \lambda_j < 0 \) when \( \beta_j \in (-1, 0) \). The intuition is straightforward; when technology suddenly improves, more of human capital is allocated to the research sector, and vice versa.

2.4.3. Growth

Next, the effects of higher uncertainty on the expected endogenous growth rate of technological progress \( g_a \equiv \mathbb{E}((dA/dt)/A) \) can be understood similarly. We have

\[
g_a = \mu H + \sum_{j=1}^{N} \lambda_j \beta_j
\]

\[
= \frac{\Lambda \mu H - \alpha \left( \rho - \mu H (\alpha + \beta) + \frac{\sigma^2}{2} (\alpha + \beta) (1 - \alpha - \beta) - \sum_{j=1}^{N} \lambda_j \left( (1 + \beta_j)^{\alpha + \beta} - 1 \right) \right) + \Lambda \sum_{j=1}^{N} \lambda_j \beta_j}{\Lambda}
\]

where \( \Lambda \equiv (1 - \alpha)(\alpha + \beta) \). Though we are primarily concerned with stochastic elements, note first that \( g_a \) is increasing in the total stock of human capital \( H \). This is what Romer (1990, p.99) shows in his seminal paper: ”an economy with a larger total stock of human capital will experience faster growth...low levels of human capital may help explain why growth is not observed in underdeveloped economies...” Here, it is the case since, with high levels of \( H \), there are more rooms for the economy to allocate the total stock of human capital to the research sector, hence stimulating R&D.

What we cannot see in the original Romer (1990) model, however, is that \( g_a \) is lowered in response to larger technology shocks \( \sigma_a \), as displayed in the lower right panel of Fig. 1. The reason is identical to that for welfare; larger technology shocks discourage the overall stock of human capital to be allocated to the research sector. With less R&D, the source of welfare \( J \), that is, technology \( A \) is reduced. As such, larger technology shocks are both welfare- and growth-reducing. Therefore, as described in Introduction, what is going on here is that, firms appear less willing to innovate in the face of the more uncertain future.

One more thing to be seen is impacts of Poisson uncertainty about technological progress. You may note that the mechanism through which Poisson uncertainty affects growth is so identical to that for welfare, that I keep the exposition short. First, we have \( \partial g_a / \beta_j > 0 \), that is, a jump of bigger size of technology is unambiguously growth-enhancing. This can again be understood by noting that \( \partial H_A / \partial \beta_j > 0 \). Put differently, a jump of bigger size cause more of human capital to be allocated to the research sector, hence stimulating R&D and accelerates technological progress. Second, the effects of higher arrival rates \( \lambda_j \) is again ambiguous. We have \( \partial g_a / \partial \lambda_j > 0 \) when \( \beta_j > 0 \) and \( \partial g_a / \partial \lambda_j < 0 \) when \( \beta_j \in (-1, 0) \). Going over, the intuition is straightforward; when technology suddenly improves, more of human
capital is allocated to the research sector and hence stimulating R&D, resulting in faster technological progress. The opposite can be interpreted in the same vein.

\[ J(K,A) \]
\[ g_a \times 10^{-3} \]

Figure 1: The impact of larger technology shocks \( \sigma_a \) on the allocation of human capital, welfare \( J(K,A) \), and expected growth rate of technology \( g_a \). Parameters are \( \alpha = 1/3, \beta = 1/3, \rho = 0.05, \eta = 1, \delta = 0.02, \) and \( \mu = 0.03 \). Without loss of generality, I normalize \( K = H = A = 1 \) for simplicity. Larger technology shocks allocate less of the total stock of human capital to the research sector \( (H_A) \) and more of it to the final-goods sector \( (H_Y) \). As a result, welfare is deteriorated and the expected growth rate of technology slows down.

2.5. Simple Simulation

This paper presumes that technological progress is better described by stochastically, not deterministically. To verify the use of the stochastic, rather than deterministic, differential equation (5), I perform some simple simulation based on Japanese data. Fig. 2 shows the index for TFP (Total Factor Productivity) in Japan before (the ”lost decade” à la Hayashi and Prescott, 2002) and after 2000. During the ”lost decade”, growth rates of TFP seem to take the ”wave” pattern, that is, they go up, down, up, and again down. Since 2000, the
actual line (with circles) basically nears the (Hodrick-Prescott filtered) trend line, except for the sharp drop in 2009 because of the global financial crisis.

Figure 2: TFP (Total Factor Productivity) in Japan between 1990 and 2010. The line with circles displays the index for TFP (normalized to 100 for 2010). The line without symbols shows the Hodrick-Prescott filtered trend. We can see that the actual TFP fluctuates around the trend line. Source: Macro-economic database AMECO at https://ec.europa.eu/info/business-economy-euro/indicators-statistics/economic-databases/macro-economic-database-ameco/ameco-database_en.

Fig. 2 makes it clear that the prediction of the deterministic growth model is not precise when shocks hit the economy, especially when the actual and trend line diverge as in the “lost decade” of 1990s in Japan. In contrast, the stochastic growth model is superior in this regard; Fig. 3 shows the simulated paths of Eq. (5) with $\sigma_a = 0\%$ (deterministic limit), 1%, 2%, and 3% (net of Poisson jumps). Qualitatively, stochastic differential equations can account for the deviation from the trend line (with $\sigma_a = 0\%$) as in the “lost decade” of Japan that we saw in Fig. 2, that is, the simulated paths fluctuates around the deterministic trend path. The bottom line of this simple simulation is that, in accounting for the growth of TFP, models with stochastic technological progress is (at least for Japan) superior to those with deterministic technological progress. Of course, the quantitative match needs more frictions, such as sticky prices, adjustment costs in investment, and so forth, but that is not the purpose of this paper and can be left for future research.

The findings of this section can be summarized as follows.

**Proposition 1.** With one parameter restriction of Xie (1991, 1994), I find the closed-form solution to the stochastic Romer (1990) model in which technological progress follows...
Figure 3: Simulation of the stochastic differential equation (5) net of jumps. Parameters are $\alpha = 1/3$, $\beta = 1/3$, $\rho = 0.05$, $\eta = 1$, $\delta = 0.02$, and $\mu = 0.03$. Without loss of generality, I normalize $H = 1$ and the initial value $A(0) = 1$ for simplicity. $\sigma_a = 0$ line is the deterministic benchmark. Other three dashed lines are simulated paths with $\sigma_a = 1\%$, $\sigma_a = 2\%$, and $\sigma_a = 3\%$. 
a combination of a controlled diffusion process and many Poisson jump processes. I show that larger technology shocks deteriorate the welfare of households and slow down the expected growth rate of technological progress. A jump of bigger size is unambiguously welfare-improving and growth-enhancing, while effects of higher arrival rates depend on the sign of jumps. Moreover, simple simulation demonstrates the importance of stochastic elements in accounting for the growth rate of TFP.

It is worth reiterating that larger technology shocks lower the expected growth rate of TFP in the stochastic Romer (1990) model. Therefore, TFP growth slows down by taking seriously the fact that not all R&D is successful into account. Although this is the theoretical finding, it might have some quantitative significance in accounting for the "lost decade" of Japan, extensively studied by Hayashi and Prescott (2002), in an *endogenous* manner.

3. Extension

In spite of some insights, the stochastic Romer (1990) model in the previous section had one possible shortcoming; consumption does not react to uncertainty terms. But consumption seems to be reduced by higher uncertainty due to precautionary saving motives. The lack of this response may come from the assumption that the accumulation of physical capital is *deterministic*. However, one can guess from (15) that, as $C$ is linearly related to $K$, the *stochastic* accumulation of physical capital may solve the irrelevance puzzle of the previous section. Therefore, in this section, I extend the model in which not only technological progress but also the accumulation of physical capital is stochastic. I then examine whether the closed-form solution is still available and consumption may react to higher uncertainty.

3.1. Risky Capital and Optimization

For my purpose, I replace Eq. (4) by the stochastic differential equation

$$dK = \eta^{\alpha+\beta-1} A^{\alpha+\beta} H_Y^{\alpha} L^{\beta} Y^{1-\alpha-\beta} dt - C dt - \delta K dt - \sigma_k K dz_k - \sum_{i=1}^{M} \beta_i K dq_i,$$

(17)

where $\sigma_k \geq 0$ is the diffusion coefficient of physical capital accumulation. $dz_k$ is the increment of a Brownian motion process such that the mean $E(dz_k) = 0$ and variance $V(dz_k) = dt$. I assume that $dz_a$ and $dz_k$ are uncorrelated in what follows. Besides, there are $M$ independent Poisson jump processes $q_i$ with the arrival rate $\lambda_i$ and a jump of size $\beta_i < 1$.

The rest of the model remains unchanged. As a consequence, a representative household maximizes its expected utility subject to two stochastic processes and the resource constraint $H_Y + H_A = H$. The HJB equation is now

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$^5$Eaton (1981) first assumes that the depreciation rate of physical capital follows a Brownian motion process. Similarly, Rebelo and Xie (1999) assume that it follows both a Brownian motion process and one Poisson jump process.
\[
\rho J(K, A) = \frac{C^{1-\phi}}{1-\phi} - \frac{1}{1-\phi} J_K Y - J_K C - J_K \delta K + \frac{J_{KK} \sigma^2_k K^2}{2 \alpha} + \mu H_A A \phi + \frac{J_{AA} \sigma^2_a A^2}{2} \\
+ \sum_{j=1}^{N} \lambda_j (J(K, (1 + \beta_j)A) - J(K, A)) + \sum_{i=1}^{M} \lambda_i (J((1 - \beta_i)K, A) - J(K, A)),
\]

where \(J_{KK} \equiv \partial^2 J/\partial K^2\). As first-order conditions (7) and (8) are identical, we can substitute these into this to get the new maximized HJB equation

\[
\rho J(K, A) = \frac{\phi}{1-\phi} J_K^2 - \frac{1}{1-\phi} J_K \delta K + (1 - \alpha)J_K^{\frac{1}{1-\phi}} \eta^{\frac{\alpha + \beta - 1}{1-\phi}} A^{\frac{\alpha}{1-\phi}} L^{\frac{\beta}{1-\phi}} K^{\frac{1 - \alpha - \beta}{1-\phi}} J_A^\frac{\alpha}{\mu} \left(\frac{\alpha}{\mu}\right)^{\frac{\alpha}{\mu}}
\]

\[
+ \mu J_A A H + \frac{J_{AA} \sigma^2_a A^2}{2} + \frac{J_{KK} \sigma^2_k K^2}{2} + \sum_{j=1}^{N} \lambda_j (J(K, (1 + \beta_j)A) - J(K, A))
\]

\[
+ \sum_{i=1}^{M} \lambda_i (J((1 - \beta_i)K, A) - J(K, A)),
\]

where there are additional two terms because of uncertainty about the accumulation of physical capital. Although it is harder to solve this more general problem, we can still solve in closed form. It can be summarized as follows.

**Theorem 2.** If we again impose the parameter restriction originally proposed of Xie (1991), that is, \(\phi = 1 - \alpha - \beta\), we can find the closed-form representation of the value function \(J(K, A)\) that satisfies TVC of the form

\[ J(K, A) = \mathcal{X} K^{\alpha + \beta} + \mathcal{Y} A^{\alpha + \beta} + \mathcal{Z}, \tag{18} \]

where \(\mathcal{Y} = \mathcal{Y}, \mathcal{Z} = \mathcal{Z}\), and

\[
\mathcal{X} \equiv \frac{1}{\alpha + \beta} \left( \rho + \delta(\alpha + \beta) + \frac{\sigma^2}{2}(\alpha + \beta)(1 - \alpha - \beta) - \sum_{i=1}^{M} \lambda_i ((1 - \beta_i)^{\alpha + \beta} - 1) \right)^{1 - \alpha - \beta}. \tag{19}
\]

Moreover, the explicit expression for \(H_A\) is still given by (16), while that for \(C\) is now

\[
C = ((\alpha + \beta)X)^{-\frac{1}{1-\alpha-\beta}} K
\]

\[
= \frac{\rho + \delta(\alpha + \beta) + \frac{\sigma^2}{2}(\alpha + \beta)(1 - \alpha - \beta) - \sum_{i=1}^{M} \lambda_i ((1 - \beta_i)^{\alpha + \beta} - 1)}{1 - \alpha - \beta} K. \tag{20}
\]

Proof. See Appendix A.
3.2. Remarks

I in turn comment on main points in Theorem 2. I refrain from repeating what I already discussed in Theorem 1.

3.2.1. Consumption

What is noteworthy here is that the optimal level of consumption (or consumption-capital ratio) reacts to uncertainty terms, that is, $\sigma_k$, $\lambda_i$, and $\beta_i$, hence solving one of shortcomings in the previous section. In particular, notice that $C$ is increasing in $\sigma_k$. In other words, higher uncertainty about physical capital accumulation induces more consumption. This finding is seemingly counterintuitive, as higher uncertainty seems to reduce consumption because of the precautionary saving motive. However, in a stochastic AK-type model, Rebelo and Xie (1999, Proposition 7), as I do here, find the closed-form solution using the parameter restriction of Xie (1991) and derive the almost identical expression for $C$ to (20). But they do not examine the impacts of $\sigma_k$ on $C$, and why $C$ can be increasing in $\sigma_k$ (this latter point gets explicit by differentiating their expression) in detail. As such, it deserves the serious discussion.

Here, there are two underlying conflicting forces. These forces reflect the counteracting influences of income and substitution effects that are frequently observed in macroeconomics. As long as agents are less risk averse, higher uncertainty boils down to an increase in income and therefore leads to an increase in consumption-capital ratio $C/K$. At the same time, higher uncertainty lowers the risk associated with savings, thereby inducing less consumption. In the current situation, the former effect always outbalances the latter, leading to a net increase in consumption. This sort of the observation is consistent with that in Turnovsky (2000, p.565). He examines the effect of an increase in the variance of wealth $W$ (the sum of physical capital $K$ and bond $B$) on the consumption-wealth ratio $C/W$ and finds that the sign depends essentially on the value of the risk aversion parameter $\phi$ (or equivalently, whether income or substitution effects dominate the other). In particular, when $\phi > 1$, the larger variance of wealth reduces $C$, when $\phi = 1$, the larger variance of wealth has no effect on $C$ (as income and substitution effect exactly offset in the case of logarithmic utility), and when $\phi < 1$ (as in this study), the larger variance of wealth induces more $C$. For this reason, higher uncertainty about the accumulation of physical capital $\sigma_k$ increases consumption $C$. As such, the finding of this paper is consistent with that of Rebelo and Xie (1999) and Turnovsky (2000), as long as $\phi < 1$.

3.2.2. Growth and Welfare

The introduction of physical capital uncertainty has no effect on the expected growth rate of technological progress $g_a$ (note that $K$ and $A$ are separable in the value function). At the same time, it affects the welfare of agents. Specifically, notice that

$$\frac{\partial X}{\partial \sigma_k} < 0.$$

---

6 Note that, due to the parameter restriction, the risk aversion parameter $\phi$ is less than one.
As $X$ is in front of $K$ in the closed-form representation of the value function $J$, this implies that higher $\sigma_k$ weakens the contribution of $K$ to $J$. In other words, larger shocks to physical capital $\sigma_k$ deteriorate the welfare of agents. Moreover, because $Y$ contains $\sigma_k$ via $X$, higher $\sigma_k$ further reduces welfare by weakening the contribution of technology $A$ to the welfare of agents $J$. Similar arguments apply to the case of Poisson jump uncertainty. In this case, we have

$$\frac{\partial X}{\partial \beta_i} < 0,$$

that is, a jump of bigger size unambiguously deteriorates welfare. In contrast, for the arrival rate, we have $\partial X/\partial \lambda_i > 0$ for a positive jump $\beta_i \in (0, 1)$, while $\partial X/\partial \lambda_i < 0$ for a negative jump $\beta_i < 0$. Clearly, "good" events are welfare-improving, and vice versa.

The findings of this section can be summarized as follows.

**Proposition 2.** With one parameter restriction of Xie (1991, 1994), I find the closed-form solution to the stochastic Romer (1990) model in which both technological progress and the accumulation of physical capital follow a mixture of a Brownian motion process and many Poisson jump processes. I show that larger shocks to physical capital deteriorate the welfare of households. Moreover, consumption is increasing in uncertainty about physical capital accumulation. As for Poisson uncertainty, a jump of big size reduces welfare, while effects of higher arrival rate depend on the sign of jumps.

4. Concluding Remarks

I first analyze the stochastic Romer (1990) model in which endogenous technological change follows a mixture of a controlled diffusion process and many Poisson jump processes. Imposing the parameter restriction of Xie (1991, 1994), I obtain the closed-form solution to the model. Using it, I demonstrate that larger technology shocks deteriorate the welfare of households and lower the expected growth rate of technological progress. The latter is especially significant; we know that the success of R&D is random. Not all R&D is successful. Taking this seriously into account, I show that such uncertainty indeed lowers the expected growth rate of TFP. Moreover, simple simulation for Japan suggests the importance of stochastic elements to account for technological progress. I then extend the baseline model in order to deal with the unsatisfactory finding that the optimal level of consumption is irrelevant to shock terms. Constructing the stochastic Romer (1990) model in which both technological progress and the accumulation of physical capital are driven by stochastic processes, I show that consumption does react to shock terms. What is more, a little counterintuitively, consumption appears to be increasing in shocks to physical capital accumulation. I account for this property in the context of conflicting forces of income and substitution effects.

The possible extension of this paper would be to find the closed-form solution without the parameter restriction of Xie (1991, 1994). As we have seen, it is very useful in that it allows us to understand the mechanism of the model in the most transparent way; in
particular, all growth and welfare analyses are possible simply by partial differentiations. But it is not possible to examine the robustness of findings when the risk aversion parameter is varied. This might be overcome by assuming the more general utility function of Duffie and Epstein (1992) that disentangles the risk aversion parameter from the intertemporal elasticity of substitution. Another direction would be finding the closed-form solution to other endogenous growth models. We now know that, beyond AK models, it may be possible to analytically investigate the properties of the stochastic Uzawa-Lucas model and Romer (1990) model. However, it is not yet obvious whether we can solve, for example, endogenous growth models of Grossman-Helpman or Aghion-Howitt in closed form under uncertainty and expand our knowledge of stochastic growth models not yet surveyed in Wälde (2011). It is certainly challenging, but the possibility is probably here to stay.

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Appendix A. Guide to the Analytical Solution

In this Appendix, I show you how to find the functional form of the value function (18). The presentation here is based on the Appendix A of Bucci et al. (2011). I postulate the tentative value function of the form

\[ J(K, A) = X K^{\theta_1} + Y A^{\theta_2} + Z, \]

where \( X, Y, Z, \theta_1, \) and \( \theta_2 \) are unknown constants to be determined. Relevant partials are

\[ J_K = \theta_1 X K^{\theta_1 - 1}, \quad J_{KK} = \theta_1 (\theta_1 - 1) X K^{\theta_1 - 2}, \quad J_A = \theta_2 Y A^{\theta_2 - 1}, \quad \text{and} \quad J_{AA} = \theta_2 (\theta_2 - 1) Y A^{\theta_2 - 2}. \]

Substituting these into the maximized HJB equation in the main text, setting \( \theta_1 = \theta_2 = \alpha + \beta, \) and imposing the parameter restriction (10), we can get the closed-form representation of the value function (18) and associated constants (12), (13), and (14) in Theorem 1, together with (19) in Theorem 2. The proof of optimality conditions requires the verification theorem. See Appendix A of Bucci et al. (2011), Appendix B of Hiraguchi (2013), or Chang (2004, Chapter 4) for details in a Brownian motion process case. For the Poisson jump case, see Sennewald and Wälde (2006) or Sennewald (2007).

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