Abstract

In the paper, by virtue of the $p$-adic fermionic integral on $\mathbb{Z}_p$, the authors consider a $\lambda$-analogue of the Changhee polynomials and present some properties and identities of these polynomials. ©2016 All rights reserved.

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1. Introduction

Let $p$ be a fixed odd prime number and let $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote respectively the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm is normally defined as $|p| = \frac{1}{p}$. Recently, degenerate Changhee polynomials $\text{Ch}_{n,\lambda}(x)$ are defined [6, p. 296] by

$$
\frac{2\lambda}{2\lambda + \ln(1 + \lambda t)} \left[ 1 + \frac{\ln(1 + \lambda t)}{\lambda} \right]^x = \sum_{n=0}^{\infty} \text{Ch}_{n,\lambda}(x) \frac{t^n}{n!}
$$

When $x = 0$, we call $\text{Ch}_{n,\lambda} = \text{Ch}_{n,\lambda}(0)$ the degenerate Changhee numbers.

It is common knowledge that the Euler polynomials $E_n(x)$ are given by

$$
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}
$$
Theorem 2.1. For polynomials, and to investigate some properties and identities of these polynomials.

Theorem 2.2. For \( f_1(x) = f(x + 1) \). Consequently, it follows from [2, p. 1256], [4, p. 994], and [5, p. 366] that

\[
\int_{\mathbb{Z}_p} (1 + t)^{x+y} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \frac{\text{Ch}_n(x)}{n!} \frac{t^n}{n!},
\]

where \( \text{Ch}_n(x) \) are called the Changhee polynomials.

We note that the Euler polynomials \( E_n(x) \) may also be represented by

\[
\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\]

The purpose of this paper is to construct a new type of polynomials, the Appell type \( \lambda \)-Changhee polynomials, and to investigate some properties and identities of these polynomials.

2. Appell type \( \lambda \)-Changhee polynomials

Assume that \( \lambda, t \in \mathbb{C}_p \) such that

\[
|\lambda t|_p < p^{-1/(p-1)} \quad \text{and} \quad (1 + \lambda t)^{x/\lambda} = e^{x \ln(1+\lambda t)/\lambda}.
\]

Now we define the Appell type \( \lambda \)-Changhee polynomials \( \mathcal{C}_h_n(x|\lambda) \) by

\[
\int_{\mathbb{Z}_p} e^{y \ln(1+\lambda t)/\lambda + xt} d\mu_{-1}(y) = \frac{2}{(1 + \lambda t)^{x/\lambda} + 1} e^{xt} = \sum_{n=0}^{\infty} \frac{\mathcal{C}_h_n(x|\lambda)}{n!} \frac{t^n}{n!}.
\]

When \( x = 0 \), we call \( \mathcal{C}_h_n(\lambda) = \mathcal{C}_h_n(0|\lambda) \) the \( \lambda \)-Changhee numbers. Note that \( \mathcal{C}_h_n(1) = \text{Ch}_n \) for \( n \geq 0 \).

**Theorem 2.1.** For \( n \geq 0 \), we have

\[
\mathcal{C}_h_n(x|\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{C}_m(\lambda)x^{n-m}.
\]

**Proof.** From (2.1), we can derive

\[
\sum_{n=0}^{\infty} \mathcal{C}_h_n(x|\lambda) \frac{t^n}{n!} = \left[ \sum_{n=0}^{\infty} \mathcal{C}_h_n(\lambda) \left( \sum_{l=0}^{\infty} \frac{x^l}{l!} t^l \right) \right] = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \mathcal{C}_m(\lambda) x^{n-m} \frac{t^n}{n!}.
\]

Equating coefficients on the very ends of the above identity arrives at the required result.

**Theorem 2.2.** For \( n \geq 0 \), we have

\[
\mathcal{C}_h_n(x|\lambda) = \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \lambda^{k-m} E_m(0) S_1(k, m) x^{n-k},
\]

where \( S_1(n, m) \) is the Stirling number of the first kind.
Proof. From Theorem 2.1 it follows that
\[
\frac{d}{dx} \mathfrak{C}_h(x|\lambda) = \sum_{m=1}^{n} \binom{n}{m} \mathfrak{C}_h(\lambda)(n-m)x^{n-m-1} = n \sum_{m=1}^{n} \binom{n-1}{m-1} \mathfrak{C}_h(\lambda)x^{n-m-1} = n \sum_{m=0}^{n-1} \binom{n-1}{m} \mathfrak{C}_{h-m-1}(\lambda)x^m = n \mathfrak{C}_{h-1}(x|\lambda).
\]
This means that \( \mathfrak{C}_h(x|\lambda) \) is an Appel sequence. Furthermore, we observe that
\[
\int_{z_0} e^{y \ln(1+\lambda t)/\lambda + xt} d\mu_1(y) = \left( \sum_{m=0}^{\infty} \lambda^{-m} \int_{z_0} y^m \mu_1(y) \frac{1}{m!} (\ln(1+\lambda t))^m \right) \left( \sum_{l=0}^{\infty} \frac{x^l}{l!} l^l \right)
= \left( \sum_{m=0}^{\infty} \lambda^{-m} E_m(0) \sum_{k=m}^{\infty} S_1(k,m) \frac{\lambda^k k^l}{k!} \right) \left( \sum_{l=0}^{\infty} \frac{x^l}{l!} l^l \right)
= \left( \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \lambda^{k-m} E_m(0) S_1(k,m) x^{n-k} \right) \frac{t^n}{n!}
= \sum_{n=0}^{\infty} \binom{n}{m} \lambda^m E_{h-m}(0) \sum_{n=m}^{\infty} \frac{x^l}{l!} l^l \left( \sum_{k=0}^{n} \binom{n}{k} \lambda^{k-m} E_m(0) S_1(k,m) x^{n-k} \right) \frac{t^n}{n!}
\]
Combining this with (2.1) yields the required identity. \( \square \)

**Theorem 2.3.** For \( n \geq 0 \), we have
\[
\sum_{m=0}^{n} \mathfrak{C}_h(m|\lambda) \lambda^{n-m} S_2(n,m) = \sum_{m=0}^{n} \binom{n}{m} B_m \left( \frac{x}{\lambda} \right) \lambda^m E_{n-m}(0),
\]
where \( S_2(m,n) \) is the Stirling number of the second kind.

**Proof.** By replacing \( t \) by \( \frac{e^x - 1}{x} \) in (2.1), we obtain
\[
\frac{2}{e^t + 1} e^{x(e^{t-1})/\lambda} = \sum_{m=0}^{\infty} \mathfrak{C}_h(m|\lambda) \frac{1}{m!} \left( \frac{e^x - 1}{x} \right)^m = \sum_{m=0}^{\infty} \mathfrak{C}_h(m|\lambda) \frac{1}{m!} \sum_{n=m}^{\infty} S_2(n,m) \lambda^n t^n \frac{n!}{n!}
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \mathfrak{C}_h(m|\lambda) \lambda^{n-m} S_2(n,m) \frac{t^n}{n!}.
\]  
Recall from [1] p. 265 that the Bell polynomials \( B_n(x) \) are generated by
\[
e^{x(e^{t-1})} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.
\]
Therefore, we acquire that
\[
\frac{2}{e^t + 1} e^{x(e^{t-1})/\lambda} = \left( \sum_{l=0}^{\infty} \frac{E_l(0)}{l!} t^l \right) \left( \sum_{m=0}^{\infty} \frac{x^m}{m!} \right) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} B_m \left( \frac{x}{\lambda} \right) \lambda^m E_{n-m}(0) \frac{t^n}{n!}.
\]
Comparing this with (2.2) leads to the required identity. \( \square \)
For \( r \in \mathbb{N} \), define the higher order \( \lambda \)-Changhee polynomials \( \mathcal{C}_{n}^{(r)}(x) \) by
\[
\int \cdots \int_{Z_p} e^{(x_1 + \cdots + x_r)\ln(1 + \lambda t)/\lambda + xt} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left[ \frac{2}{(1 + \lambda t)^{1/\lambda} + 1} \right]^{r} e^{xt} = \sum_{n=0}^{\infty} \mathcal{C}_{n}^{(r)}(x|\lambda) \frac{t^n}{n!}, \tag{2.3}
\]
When \( x = 0 \), we call \( \mathcal{C}_{n}^{(r)}(\lambda) = \mathcal{C}_{n}^{(r)(0|\lambda)} \) the higher order \( \lambda \)-Changhee numbers.

**Theorem 2.4.** For \( n \geq 1 \), we have
\[
\mathcal{C}_{n}^{(r)}(x|\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{C}_{m}^{(r)}(\lambda) x^{n-m} \quad \text{and} \quad \frac{d}{dx} \mathcal{C}_{n}^{(r)}(x|\lambda) = n \mathcal{C}_{n-1}^{(r)}(x|\lambda).
\]

**Proof.** This follows from the observation that
\[
\sum_{n=0}^{\infty} \mathcal{C}_{n}^{(r)}(x|\lambda) \frac{t^n}{n!} = \left[ \frac{2}{(1 + \lambda t)^{1/\lambda} + 1} \right]^{r} e^{xt} = \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{n} \binom{n}{m} \mathcal{C}_{m}^{(r)}(\lambda) x^{n-m} \right] \frac{t^n}{n!}.
\]

Recall from [S p. 12] that the higher order Euler polynomials \( E_{n}^{(r)}(x) \) may be represented by
\[
\int \cdots \int_{Z_p} e^{(x_1 + \cdots + x_r + x)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \left( \frac{2}{e^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_{n}^{(r)}(x) \frac{t^n}{n!}.
\]
When \( x = 0 \), we call \( E_{n}^{(r)}(0) \) the higher order modified Euler numbers.

**Theorem 2.5.** For \( n \geq 0 \), we have
\[
\mathcal{C}_{n}^{(r)}(x|\lambda) = \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \lambda^{k-m} E_{m}^{(r)}(0) S_1(k,m) x^{n-k}.
\]

**Proof.** We observe that
\[
\int \cdots \int_{Z_p} e^{(x_1 + \cdots + x_r)\ln(1 + \lambda t)/\lambda + xt} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
\]
\[
= \left( \sum_{m=0}^{\infty} \int \cdots \int_{Z_p} (x_1 + \cdots + x_r)^m d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \frac{[\ln(1 + \lambda t)]^m}{m! \lambda^m} \right) \sum_{l=0}^{\infty} \frac{x^l}{l!}
\]
\[
= \left( \sum_{m=0}^{\infty} \lambda^{-m} E_{m}^{(r)}(0) \sum_{k=m}^{\infty} S_1(k,m) \frac{\lambda^k k!}{k!} \right) \left( \sum_{l=0}^{\infty} \frac{x^l}{l!} \right)
\]
\[
= \left( \sum_{k=0}^{\infty} \left( \sum_{m=0}^{k} \lambda^{k-m} E_{m}^{(r)}(0) S_1(k,m) \frac{t^k}{k!} \right) \left( \sum_{l=0}^{\infty} \frac{x^l}{l!} \right) \right)
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \lambda^{k-m} E_{m}^{(r)}(0) S_1(k,m) x^{n-k} \right) \frac{t^n}{n!}.
\]
Combination of this identity with \((2.3)\) results in the required identity. □

**Theorem 2.6.** For \( n \geq 0 \), we have
\[
\sum_{m=0}^{n} \lambda^{-m} \mathcal{C}_{m}^{(r)}(x|\lambda) S_2(n,m) = \sum_{m=0}^{n} B_{m} \left( \frac{x}{\lambda} \right) \lambda^m E_{n-m}^{(r)}(0).
\]
Proof. Substituting $e^{t e^{-1}} / \lambda$ for $t$ in (2.3) gives

$$
\left( \frac{2}{e^t + 1} \right)^r e^{(e^{t e^{-1}} - 1) / \lambda} = \sum_{m=0}^{\infty} \mathcal{Ch}_m^{(r)} (x | \lambda) \frac{1}{\lambda^m} \frac{1}{m!} (e^{\lambda t} - 1)^m
$$

$$
= \sum_{m=0}^{\infty} \mathcal{Ch}_m^{(r)} (x | \lambda) \frac{1}{\lambda^m} \sum_{n=m}^{\infty} S_2(n, m) \frac{\lambda^n t^n}{n!} = \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{n} \lambda^{n-m} \mathcal{Ch}_m^{(r)} (x | \lambda) S_2(n, m) \right] \frac{t^n}{n!}.
$$

On the other hand,

$$
\left( \frac{2}{e^t + 1} \right)^r e^{(e^{t e^{-1}} - 1) / \lambda} = \left[ \sum_{l=0}^{\infty} E_l^{(r)} (0) \frac{t^l}{l!} \right] \left[ \sum_{m=0}^{\infty} B_m \left( \frac{x}{\lambda} \right) \frac{\lambda^m t^m}{m!} \right] = \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{n} B_m \left( \frac{x}{\lambda} \right) \frac{\lambda^m t^m}{m!} \right] \frac{t^n}{n!}.
$$

The required result thus follows.

Remark 2.7. This paper is a slightly modified version of the preprint [7].

References